Chapter 5

Gravity and Moving Objects

fancy
geodesics in
space-propertime

Having studied 'static' acceleration to some detail, the step towards acceleration experienced by moving objects seems to be relatively easy. However, we will soon see that space curvature complicates things considerably. Fortunately the results are relatively simple in the case of Schwarzschild coordinates. We will consider pure radial movement in this coordinate system first.

5.1 Gravity and radial movement

As has been seen many times before, radial movement means that the space-propertime path is not perpendicular to the space axis. It makes an angle

$$\varphi = \arcsin(v/c)$$

with the *time axis*, where v is the locally measured radial velocity.

In the static case of the previous chapter, the centre of curvature for the geodesic was situated on the space-hyperspace plane.indexspace-hyperspace plane This is not so for the case with a radial starting velocity. The line from the origin to the centre of curvature now makes the same angle as above, i.e.

$$\varphi = \arcsin(v/c)$$

with the *space axis*.

As shown in figure 5.1, the original (static) geodesic centre of curvature (cc) is 'raised' perpendicular to the space-hyperspace plane, to a new position cc'. This action increases the radius of curvature of the geodesic and thereby decreases the radial acceleration.

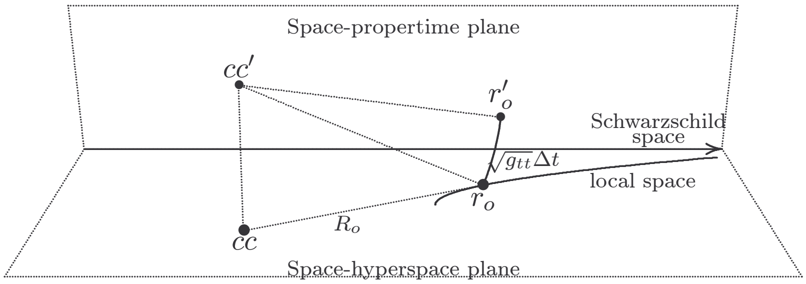

Figure 5.1: Geodesic movement through space-hyperspace-propertime for an object with a positive radial velocity. The centre of geodesic curvature (gc) is 'raised' normal to the space-hyperspace plane to the point gc', to give a larger effective radius of curvature. This reduces the effective radial acceleration.

The locally measured radial acceleration is shown in appendix C to be

$$a = -g_{rr}\frac{GM}{r^2}(1 - \frac{v_\varphi^2}{c^2}),$$

where v_φ is the locally measured radial velocity.

This is a quite understandable combination of the static acceleration ($-g_{rr}\frac{GM}{r^2}$) of the previous chapter and the (square of) the velocity time dilation of special relativity, $\sqrt{1 - v^2/c^2}$. It is clear that the local radial velocity diminishes the static acceleration.

This is easy to comprehend, in the sense that when an object is accelerated to the local speed of light, it cannot be accelerated any further. Hence radial acceleration becomes zero.

Due to space curvature, the coordinate radial acceleration requires a pretty tricky transformation of the local acceleration. This is shown in appendix C to result in the following expression:

$$a_r = -\frac{GM}{r^2}(g_{tt} - 3g_{rr}\frac{v_r^2}{c^2}),$$

where v_r is the coordinate radial velocity. Note that $-GM/r^2$ is the Newton gravitational acceleration, which is reduced in magnitude firstly by g_{tt} (which is less than unity) and secondly by an effective positive acceleration component

$$\frac{GM}{r^2} \times 3g_{rr}\frac{v_r^2}{c^2}$$

that is dependant upon space curvature g_{rr} and radial velocity v_r.

It is somewhat like when the radial speed of an aircraft is measured by a Doppler-radar, while the aircraft is flying at a constant ground speed and

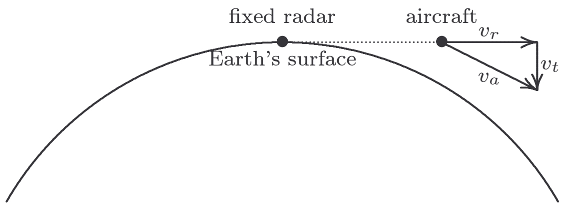

Figure 5.2: Doppler-radar measurement of the radial speed of an aircraft, flying at constant altitude and ground speed, measured near the horizon. If the true ground speed of the aircraft is represented by vector v_a, the radar will measure a radial speed of $v_r = \sqrt{v_a^2 - v_t^2}$. For illustrative purposes, the refraction of the radar signal through the atmosphere is ignored.

altitude, in a direction directly away from the radar. Due to the curvature of Earth, the measured radial velocity will decrease (see figure 5.2).

So judged purely by the measured radial speed, the aircraft seems to be decelerating. In the case of our distant observer, the same type of effect is operating due to space curvature. The effect is discussed in detail in appendix C.

In order to get a feeling for the magnitude of this opposing acceleration, let us consider a meteorite hitting the Earth's surface radially at escape velocity, about 11 km/s (ignoring the atmosphere).

For Earth, we can simply work with $-GM/r^2 = 1$g and express our results in g's. The speed of 11 km/s translates to $v_r \approx 3.7 \times 10^{-5}\ c$. This gives an opposing acceleration

$$a_{r(opp)} \approx -1 \times 3 \times (3.7 \times 10^{-5})^2 \approx -4 \times 10^{-9} \text{ g}.$$

The negative sign means 'upwards', since 'g' works downwards. This acceleration of -4 nano-g is four times the opposing acceleration caused by the gravitational redshift alone (the static relativistic effect).

However, if that 'meteorite' is a muon particle hitting the Earth's surface at $v_r = 0.99c$, the opposing acceleration works out to

$$a_{r(opp)} \approx -1 \times 3 \times 0.99^2 \approx -3\text{g}.$$

So even ignoring atmospheric drag and other effects, the muon will be *decelerating* at 1 - 3g = -2g just before it hits the surface. Yes, -2g is correct! This is some serious braking, especially for a particle that is being 'pulled' by Earth's gravity.

But remember that it is the acceleration that is (hypothetically) measured in the coordinate system, i.e. by a distant observer. Due to the space curvature between the muon and the observer, the apparent opposing acceleration simply 'overwhelms' the normal Newtonian acceleration.

Engineers are trained to question "funny" results. So how do we get some confidence in the "funny" result of the calculation in question? The answer lurks in the Schwarzschild coordinate velocity of light. In a purely radial direction, the speed of light is

$$c_{rad} = (1 - \frac{2GM}{rc^2})\ c.$$

This means it is slower near a massive body than far from it. If, using this equation, the path of a photon is modeled in a radial direction and the acceleration extracted from the data, the result is an apparent coordinate acceleration of

$$a = 2GM/r^2$$

near the surface of Earth. Contrast this with the normal Newtonian acceleration of

$$a = -GM/r^2 = 1\text{g}.$$

So for Earth's surface, radial light seems to accelerate at -2g in the coordinate system. There is a -3g acceleration opposing the Newtonian value, as measured in the coordinate system.

The muon, moving at very close to the speed of light will suffer roughly the same effect. Otherwise it might eventually have exceeded the speed of light.

One must however remember that for the local observer, momentarily stationary as the test object passes, the observed radial acceleration will be quite different:

$$a_{loc} = \frac{-GM}{r^2}\sqrt{g_{rr}}\left(1 - \frac{v_{loc}^2}{c^2}\right),$$

where v_{loc} is locally measured radial velocity.

The factor $\sqrt{g_{rr}}$ (space curvature) enhances local radial acceleration and the radial velocity factor diminishes local radial acceleration. The local radial acceleration can never become positive, because $v_{loc} < c$. The local acceleration approaches zero if the local radial velocity approaches the speed of light.

For the case of the muon hitting Earth's surface, we are approximately local observers* and if we assume that there is no space curvature between us

*Strictly, we should be free-falling and momentarily stationary.

and the muon's point of impact, we will measure the muon's acceleration as

$$a_{loc} \approx 1 \times (1 - 0.99^2) \approx 0.02\ \text{g}.$$

This means that we will observe the muon (atmospheric drag and other effects ignored) to accelerate at only 0.02g (downwards) when it hits the surface. Muons travel at so close to the speed of light that Earth's gravity can hardly accelerate them any more.

Figure 5.3 shows a plot of the gravitational acceleration measured by a distant and a local observer respectively, for an object falling to the event horizon of black hole of a million solar masses, starting from rest at a large

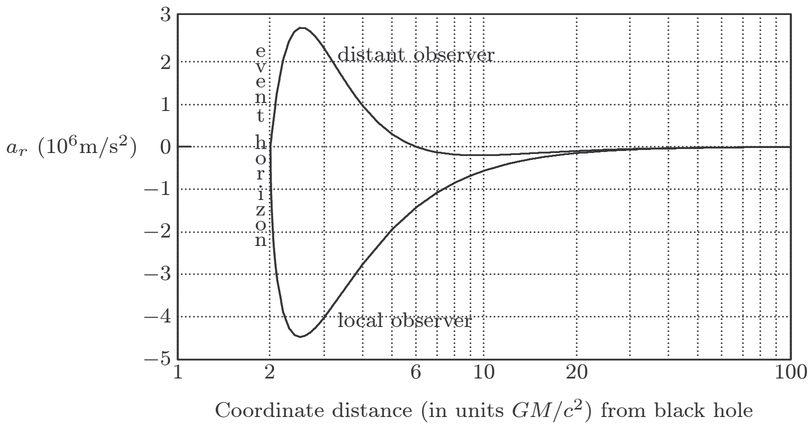

Figure 5.3: Radial acceleration of an object free-falling at (minus) escape velocity towards a black hole of a million solar masses, as measured by a distant observer (top curve) and the transformation of that acceleration to the reference frame of a local observer (bottom curve).

distance and moving towards the hole at very close to (negative) escape velocity.

The acceleration in the distant observer's frame peaks (negatively) at $a_r \approx -2 \times 10^5$ m/s^2 around $r = 9.5GM/c^2$, becomes zero at $r = 6GM/c^2$ and reaches a maximum positive value of $a_r \approx 2.7$ million m$/s^2$ around $r = 2.5GM/c^2$. Near the event horizon the observed acceleration tends to zero, due to both g_{tt} and v_r tending to zero.*

*Radial escape velocity in the distant frame is given by $v^2_{r(esc)} = g^2_{tt} \times 2GM/r$.

To the locally stationary observer however, the acceleration stays negative and peaks at $a_r \approx -4.4$ million m$/s^2$ around $r = 2.5GM/c^2$. When approaching the event horizon, the local acceleration also tends to zero as the local velocity approaches the speed of light.

Precisely at the event horizon, the equation for local radial acceleration diverges due to $g_{rr} \to 1/0$. However, $(1 - v^2/c^2) \to 0$ at a faster rate, so the acceleration tends to zero.

5.2 Gravity and transverse movement

In order to picture transverse movement in the space-hyperspace-propertime domain, one needs at least 4 dimensions in your diagram, because you need 2 normal space dimensions—one space dimension for the transverse movement and one space dimension for gravity to work in.

Since 4 dimensional drawings are not possible, it is usual to leave out either the propertime dimension or the hyperspace dimension. Fortunately, for visualizing purely transverse geodesics, the hyperspace dimension is not all that important, because the test object remains at a constant distance from the primary mass and experiences constant space curvature.*

*The space curvature is only precisely constant if the primary mass is spherically symmetrical and non-rotating.

The circular orbit is not the spacetime geodesic; one must view the time component of the geodesic as well. The circular orbit of a massive object transcribes a helix in spacetime. Figure 5.4 pictures a small segment of such a geodesic of a circular orbit.

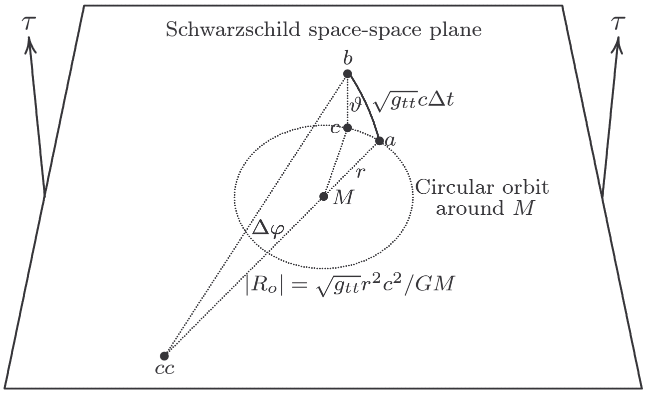

Figure 5.4: Geodesic movement through space-propertime for an object in a circular orbit around a mass M. The circle segment cc-a-b is tilted by an angle $\vartheta = \arcsin(v_o/c)$ relative to the normal to the space-space plane, where v_o is the locally measured circular orbital velocity. The angular movement of the geodesic is $\Delta\varphi = \sqrt{g_{tt}}c\Delta t/|R_o|$. The projection c (of point b) onto the space-space plane is on the circular orbit.

It is not a trivial exercise to rigorously find relativistic gravity's centripetal acceleration as measured relative to a straight line in Euclidean space, so the details will be discussed in Appendix C. Here it will only be loosely discussed as the centrifugal acceleration measured by a local observer.

First, we look at the Newtonian centrifugal acceleration v_o^2/r, where v_o is the circular orbital velocity. This is the velocity where the centrifugal acceleration precisely balances the Newtonian gravitational acceleration $-GM/r^2$, or

$$\frac{v_o^2}{r} - \frac{GM}{r^2} = 0.$$

It can be shown that the locally stationary observer will, for the same circular orbit, obtain a relativistic balance of a accelerations

$$\frac{g_{tt}v_o^2}{r} - \frac{GM}{r^2} = 0,$$

where v_o is the circular orbit velocity as measured by the local observer. After a bit of algebraic juggling,* this can be shown to be equivalent to

*Replace g_{tt} with $1 - 2GM/(rc^2)$, multiply out and collect GM/r^2 terms on the right.

$$\frac{v_o^2}{r} = \frac{GM}{r^2}(1 + 2v_o^2/c^2),$$

which, after being transformed to the frame of the distant observer gives the relativistic centripetal acceleration for a circular orbit as

$$a_{cp} = \frac{-GM}{r^2}(g_{tt} + 2v_t^2/c^2),$$

where $v_t^2 = g_{tt}v_o^2$ is the transverse (orbital) velocity as measured by a distant observer.

The above treatment is only valid for circular orbits, but the result happens to be valid for any value of purely transverse velocity, as is proven in Appendix C, where the centripetal acceleration at the periapsis or apoapsis of an elliptical orbit is derived.

An interesting and important observation from the last equation is the following: since $g_{tt} = 1 - 2GM/(rc^2)$, it is clear that for $v_t^2 > GM/r$, a_{cp} is larger than standard Newtonian acceleration ($-GM/r^2$). For $v_t^2 < GM/r$, a_{cp} is smaller than Newtonian acceleration.

At $v_t^2 = GM/r$ the opposing and additive components cancel each other, representing a circular orbit, and the centripetal acceleration is Newtonian ($-GM/r^2$).

This tells us that a perfectly circular orbit 'looks' Newtonian when measured by a distant observer. Relativistic effects do not show up in perfectly circular orbits measured form afar (i.e. in Schwarzschild coordinates).

Non-circular orbits, on the other hand, suffer increased centripetal acceleration when they are inside the equivalent circular orbit radius (i.e., near the perihelion of a planet), where the transverse velocity is greater than the circular orbital velocity.

The opposite happens when the orbit goes outside of the equivalent circular orbital radius. These are contributing factors to the perihelion shift of a planet with a fairly eccentric orbit, like Mercury, as will be discussed in the next chapter.

Looking at the centripetal acceleration equation, one can say that we have an opposing acceleration due to the gravitational redshift alone, of

$$a_{redsh} = \frac{2(GM)^2}{r^3c^2}.$$

For Earth's surface, we know this from a previous calculation as about -1 nano-g .

Then there is an 'additive' acceleration of

$$a_{add} = \frac{-2GM}{r^2}\frac{v_t^2}{c^2},$$

due to the transverse velocity alone, working with the Newtonian acceleration.

If we have a high speed particle, say the muon again, this time momentarily moving purely transverse (i.e., horizontally) close to Earth's surface at $v_t =$

$0.99c$, we can calculate the additive acceleration as follows: since $-GM/r^2$ equals 1g, the additive acceleration must be:

$$a_{add} = 2 \times 0.99^2 \text{ g} \approx 2 \text{ g},$$

again ignoring atmospheric drag and other effects.

This means that the muon will appear to be accelerated towards Earth's centre at about 3g (the normal +1g plus the 2g additive). This approximate value is equally valid for local and distant observers. As an exercise, the reader may perhaps contemplate on why this is so.*

*Hint: Purely transverse velocity does not involve spacetime curvature.

5.3 Radial and transverse velocities combined

In the previous sections, we have treated purely radial and purely transverse movement only. So what happens if both radial and transverse movements are present? The geodesic movement will then cause a non-circular orbit of some sorts.

It is possible, by the same type of arguments that we used for purely transverse or purely radial movement, to calculate the spacetime geodesic in four dimensions. The projection of the geodesic onto a flat 2-dimensional space surface then gives the relativistic orbit.

It is however quite a messy operation, involving tricky tensor algebra. A more 'engineering-like' method is to try and work out the equivalent 'Newtonian' acceleration by combining the two accelerations that we obtained before (for radial and transverse movements respectively).

This is a relatively simple operation. The accelerations that we have worked out so far all works in the radial direction and we can add them up in a Newtonian manner. The resultant radial acceleration, as measured in the distant (coordinate) frame, is then

$$a_r = \frac{-GM}{r^2}\left(g_{tt} - 3g_{rr}\frac{v_r^2}{c^2} + 2\frac{v_t^2}{c^2}\right),$$

where v_r and v_t are respectively the radial and transverse velocity components measured in the distant frame.

So can we go and calculate the trajectory that a particle will follow in space by using this radial acceleration in a Newtonian manner? Not quite, because we are not done yet.

Rather surprisingly, there are additional transverse acceleration components in relativistic gravity. It should not be surprising, though, because Newton's trajectories in gravitational field undergo transverse accelerations in most cases.

An aside... When the author reached this point in developing the quasi-Newtonian interpretation, he promptly plugged the radial acceleration equation into a program for calculating and plotting orbits. In strong gravitational fields the orbits were way off the mark, when compared to standard relativistic solutions (discussed in the next chapter). So back to the drawing board...

Think about the ball rolling down a slope. The gravitational force works vertically downwards, yet the ball picks up speed down the slope. That acceleration contains both radial and transverse components.

Both these components are modified in curved spacetime. Hence, there is at least one transverse component to factor into a quasi-Newtonian calculation. It will become clear later that this must be an opposing acceleration in the transverse direction.

To return to the ball down the slope example: the ball will pick up transverse speed slower than expected. If the ball is rolled up the slope, it will lose less speed than expected. In the low speed, weak gravity case, the effect will be negligible, but not so at relativistic velocities.

The magnitude of the apparent opposing transverse acceleration can be found in a similar way as for purely radial movement. The apparent opposing radial acceleration was previously found to be

$$a_{r(opp)} = \frac{3GM}{r^2} g_{rr} \frac{v_r^2}{c^2}.$$

As discussed in appendix C, the transverse component has the same form, but with a factor 2 instead of 3 and with v_r^2 replaced by the product $v_r v_t$, i.e.,

$$a_{t(opp)} = \frac{2GM}{r^2} g_{rr} \frac{v_r v_t}{c^2}.$$

If we are close to a black hole and the product $v_r v_t \neq 0$, then g_{rr} plays a significant role. In the low gravity of Earth, where $g_{rr} \approx 1$, significant velocities tend to dominate the action.

As an example, let our muon particle again momentarily move at a speed of about 0.99c, but this time at an angle of 45 degrees off the radial towards Earth, so that $v_r \approx -0.7c$ and $v_t \approx 0.7c$. In the absence of atmospheric drag and other effects, the opposing transverse acceleration will be

$$a_{t(opp)} \approx -2 \times -0.7 \times 0.7 \approx 1\text{g}.$$

The starting negative sign is because g is a negative (-9.8 m/s^2) acceleration.

This is a very significant acceleration in the negative transverse direction, i.e., against the direction of transverse movement. For comparison, let us also calculate the total radial acceleration of the muon for this scenario, i.e.,

$$a_r \approx [1 - 3 \times (-0.7)^2 + 2 \times 0.7^2] \approx 0.6\text{g}.$$

In Newtonian dynamics, $a_{t(opp)} = 0$ and $a_r = 1$g, so one can expect that there will be a significant difference between the relativistic track and the Newtonian track of the muon.

The relativistic radial acceleration is smaller than the Newtonian value, but the opposing transverse acceleration will cause the muon to curve a little more strongly towards the centre of Earth than what Newton would have predicted.

> To understand why, think about what would happen if the muon's transverse velocity approaches zero (which is unlikely, but it illustrates the point).

So much for the rather weak gravitational field of planet Earth. Next, we will work through an example in a strong gravitational field. Let a distant observer measure the accelerations of our muon, instantaneously at one Earth radius from a black hole with a mass of a hundred million (10^8) Earths.

This gives $g_{tt} = 0.861$ and the Newtonian radial acceleration will by definition be 10^8g.

Assume instantaneous (local) velocity components of $v_r = 0.7c$ and $v_t = 0.7c$ (both positive in this case). The velocities transform to distant (coordinate) velocity components

$$v_r = 0.7c \times 0.861 \approx 0.6c$$

and

$$v_t = 0.7c \times \sqrt{0.861} \approx 0.65c.$$

> If the coordinate velocity components were higher by as much as 0.1%, the local velocity would have exceeded the speed of light, which is not allowed.

The radial acceleration comes out as

$$a_r \approx 10^8 \left(0.861 - 3\frac{(0.6)^2}{0.861} + 2 \times 0.65^2\right) \approx 4.4 \times 10^7\text{g}.$$

The opposing transverse acceleration works out to be

$$a_{t(opp)} \approx -2 \times 10^8 \, \frac{0.6 \times 0.65}{0.861} \approx -9.1 \times 10^7\text{g}.$$

The negative result means it is a positive acceleration, in-line with the transverse movement (again because we work with g = -9.8 m/s^2).

One can expect that the lower radial acceleration, coupled with a significant transverse acceleration will result in a trajectory dramatically different from a Newtonian trajectory. Just how dramatic will become clear in the next chapter, on orbits.

The typical engineer's question to the above is: how can a purely radial gravitational 'force' produce an acceleration that is normal to the force? It was loosely answered before. Appendix C attempts to answer this question in a rigorous way. The following is another 'loose' view of the effect.

Newton trajectories also cause (non-zero) transverse speed to either increase or decrease, depending whether the trajectory is in-falling or out-falling. The conservation of energy demands it. In the curved spacetime world of general relativity, energy conservation works slightly differently and this causes additional accelerations.

A second common question is: if an observer could ride with the muon, would the opposing accelerations be perceived as 'opposing forces'? The answer is no; in the absence of other external forces, the muon (and hypothetical observer) follows a spacetime geodesic and would not experience any forces.

> An observer might perhaps experience tidal gravity forces, which will be a squeeze and a stretch. More about that in a later chapter.

Now for a crucial question: are these opposing and additive accelerations real, or are they just artefacts of measurement? The simplest, yet useless answer is that they are a bit of both. They are real in the sense that they can be measured—the paths of fast moving objects in a gravitational field is different from Newtonian predictions.

On the other hand, the relativistic accelerations differ, depending on who is making the measurements. The local and distant observers get different answers when they measure the acceleration of the same object moving in the same gravitational field.

This is because their measurement rods are of different length (or of different orientation in curved space) and their clocks tick at different rates. All these are most notable when strong gravitational fields are involved.

In the next chapter it will be shown that the relativistic gravitational accelerations can be used in a quasi-Newtonian way to construct orbits. Such orbits are virtually indistinguishable from the orbits produced by the more formal relativistic orbital equations.

5.4 Newton's gravity and the speed of light

Although Newton surely never contemplated that the speed of light is infinite, it is interesting to note that if we take relativistic gravity and set $c \to \infty$, it reduces to Newton's gravity. First, look at the metric coefficients g_{tt} and g_{rr}

$$g_{tt} = 1 - \frac{2GM}{rc^2}.$$

It reduces to unity when $c \to \infty$, and since $g_{rr} = 1/g_{tt}$, so does g_{rr}. Taken only this into account, the gravitational accelerations reduce to:

$$a_r \quad \to \quad -\frac{GM}{r^2}\,(1 - 3\frac{v_r^2}{c^2} + 2\frac{v_t^2}{c^2})$$

$$a_t \rightarrow 2\frac{GM}{r^2}\frac{v_r v_t}{c^2}.$$

It is fairly obvious that all the speed ratios (v/c) approach zero when $c \rightarrow \infty$, (for all speeds less than infinite, that is) so that what we have left is the Newtonian acceleration

$$\begin{aligned} a_r &\rightarrow -\frac{GM}{r^2} \\ a_t &\rightarrow 0. \end{aligned}$$

It is also interesting to note that the Schwarzschild radius $r_S = 2GM/c^2$ reduces to zero when $c \rightarrow \infty$, so no black holes, only (naked)singularities at the centre. Since the acceleration would be Newtonian, one can expect that all orbits would be Newtonian.

All the above is roughly the same thing as leaving the value of c at its normal value and stating that Newton's gravity holds in the weak field $(GM \ll rc^2)$, low speed $(v \ll c)$ domain.

If Einstein was right, it seems that should $c \rightarrow \infty$, the weak field, low velocity domain rules everywhere. But then, as far as we know, there was never a time when the speed of light approached infinity.

5.5 Summary of relativistic acceleration

In this chapter, the static relativistic acceleration has been developed further into firstly, acceleration with pure radial movement, secondly, acceleration with pure transverse movement and then radial acceleration with both radial and transverse movements.

It was shown that radial movement decreases radial acceleration in both the local reference frame and the coordinate reference frame. For the local frame, it is just a velocity time dilation type effect. For the coordinate frame, the velocity time dilation is enhanced by a space curvature effect.

The most interesting effect of this chapter is likely to be the additional (and conditional*) component of acceleration in the transverse direction. We have seen how high velocities with both radial and transverse components influence the acceleration that particles experience. This is most severe in strong gravity fields, but it is even significant in the weak field of Earth.

*Recall that the opposing transverse acceleration is a function of the product of radial and transverse velocity. If either is zero, it vanishes.

Finally, the fact that if the speed of light was infinite, relativistic gravity would reduce to Newton's gravity was loosely discussed. This could make one wonder if the 'speed of gravity' is not infinite. We will discuss this at the end of the next chapter (orbital dynamics).