Chapter 8

Some tests of general relativity

perihelion shift, deflection,
time delay, geodetic effect,
dragging of inertial frames

In this chapter, we will discuss the three most well known tests of Einstein's general theory of relativity, i.e., the perihelion shift of Mercury, the bending of light by the Sun's gravity and the time delay of light passing close to the Sun. There have been others, which will be mentioned in later chapters, but the three discussed here are all relatively easily understood, using the knowledge gained in the previous chapter.

If you have not yet read Clifford Will's bestseller on tests of general relativity, "Was einstein Right?" [Will(b)], it is highly recommended. It is an excellent nontechnical description of most of the tests performed up to the mid 1980s, and some that are yet to be performed.

In this chapter, the reader will be filled in with some of the technical details that professor Will had to leave out, probably on the insistence of his publisher, who most likely wanted a popular bestseller!*

*The heavy technical details are found in his book [Will(a)].

8.1 The perihelion shift of Mercury

By the time Einstein was working on his general theory, the anomalous shift of the perihelion of Mercury was well known. It was anomalous, because when astronomers carefully applied Newton's theory of gravity to the orbit of Mercury, and they compensated for all the known effects of perturbation caused by the other planets on the orbit, there was a 'residue' of about 43 arcseconds per century.

Einstein was well aware of this and has used it to some degree to 'test' his theory. From the full relativistic orbital equation

$$\frac{d^2u}{d\theta^2} = \bar{M}/\tilde{L}^2 - u + 3\bar{M}u^2,$$

it takes just a bit of mathematical sweat to prove that for nearly circular orbits, i.e., $e \approx 0$, the perihelion shift per orbit is

$$\Delta\theta_p \approx \frac{2\pi}{\sqrt{1 - 6\bar{M}/r_o}} - 2\pi,$$

where r_o is the average orbital radius around mass $\bar{M}$. This says that instead of sweeping out an angle 2π between two successive points of closest approach (the perihelion of a planetary orbit), the orbit sweeps out an angle 2π divided by a 'modified time dilation' $\sqrt{1 - 6\bar{M}/r_o}$, making $\Delta\theta_p > 0$, always.

Let us plug actual values into the equation. The mass of the Sun is about 1477m (geometrically) and the average orbital distance of Mercury from the Sun is about 5.79 $\times 10^{10}$m, giving a perihelion shift per orbit of

$$\Delta\theta_p \approx \frac{2\pi}{\sqrt{1 - 6 \times 1477/(5.79 \times 10^{10})}} - 2\pi \approx 4.88 \times 10^{-7} \text{ radians.}$$

Mercury completes about 415 orbits every Earth century, giving the answer per century as about 0.2 mrad., or about 41 arcseconds.

This falls short from the actual measurement by 2 arcseconds, which is due to the fact that Mercury has a pretty elliptical orbit ($e = 0.2056$), making the approximate equation slightly inaccurate. It does however illustrate the point. In the later chapter, about the Post-Newtonian Formalism, we will discuss a more accurate approximation.

8.2 The deflection of light by the Sun

Einstein completed his general theory during World War I, so there was not much change to quickly verify his prediction that the Sun will bend the light rays passing close to it's surface (called a 'Sun-grazing ray') by 1.75 arcseconds. Shortly after the war, in May 1919, British astronomer Sir Arthur Eddington succeeded to measure the deflection of light from stars during a total eclipse of the Sun, on an island off the coast of (then) Spanish Guinea.

This particular test was not very accurate, although Sir Arthur claimed a value of 1.60 ± 0.31 arcseconds. This was at the time close enough to Einstein's prediction, to show that there is more to light deflection than what any 'quasi-Newtonian' theory could come up with. If one assume the corpuscular theory for light, then those particles, traveling at the speed of light, would according to Newton's gravity, suffer a deflection of 0.875 arcseconds if they graze the surface of the Sun—precisely half of what Einstein predicted![Faber].

It was later discovered by theorists that any theory of gravity that is compatible with equivalence principle (equivalence of acceleration and gravity), will automatically predict a light deflection of 0.875 arcseconds for a Sun-grazing ray. The other 0.875 arcseconds comes from the curvature of space [Will(b)].

So what does the deflection formula look like? Starting with the orbital equation for light,

$$\frac{d^2u}{d\theta^2} = -u + 3\bar{M}u^2,$$

and using successive approximations, one obtains a rather simple formula for a small deflection of light in the relatively weak gravitational field at the surface of the Sun:*

*The approximation does not hold for large deflections in strong gravitational fields. It is then better to solve the equation numerically.

$$\Delta\phi \approx \frac{4\bar{M}_{Sun}}{r_{Sun}}.$$

The radius of the Sun is about 6.96×10^8m, so that the deflection works out to [Faber]

$$\Delta\phi \approx \frac{4 \times 1477}{6.96 \times 10^8} \approx 8.49 \times 10^{-6} \text{ radians } \approx 1.75 \text{ arcseconds.}$$

This prediction has later (in the 1970s) been confirmed to the 1% level, using long baseline radio interferometry [Will(a)]. The radio waves were coming from quasars that, as the Earth orbits the Sun, pass close to the Sun once a year.

8.3 The Shapiro time delay of light

As far as we can ascertain, the speed of light in free space is one of the universal constants. It does not necessarily mean that the speed of light is the same everywhere—we know that it is slower in air than it is in space and also slower in glass than it is in air.

Gravity also has an effect on the speed of light in a certain way. That effect is not locally measurable though. If an inertial observer, momentarily stationary near a massive object, makes a speed of light measurement, the standard value of $c = 2.99792458 \times 10^8\, m/s$ will be obtained. In geometric units the locally measured speed of light is unity of course. So what is the effect we are after?

8.3.1 The effect of gravitational time dilation

The closer a clock is to a gravity generating mass, the slower it runs. How does that affect the measurement of the speed of light? If distances remained the same for local and distant observers, then the local observer must get a speed of light larger than c. We have seen before that gravity contracts space by the same factor that it dilates time.

If we could observe a pulse of light from a large distance while it moves close to and transversely relative to a massive object, we would perceive that it has slowed down by a factor $\sqrt{g_{tt}}$, due to gravitational time dilation (or gravitational redshift, if you like). The best way to put it is that the local transverse speed of light transforms to the distant reference frame by a factor $\sqrt{g_{tt}}$, just like any other transverse velocity.

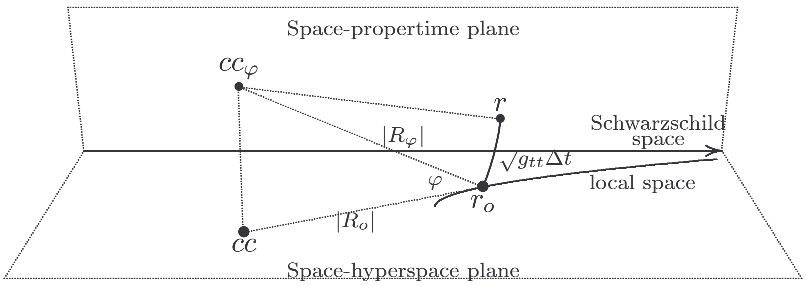

Figure 8.1: A repeat of figure 5.1 (geodesic movement) of the previous chapter. If $\varphi \to \pi/2$, i.e. the radial velocity $\dot{r} \to 1$, then R_φ tends to infinity and the space-propertime path ($r_o - r$, length $\sqrt{g_{tt}}\Delta t$) of a radially moving object will tend to coincide with the local space curve. The projection of $r_o - r$ onto Schwarzschild space will then have a length $\sqrt{g_{tt}}\Delta t$, due to the 'angle' between local space and Schwarzschild space.

8.3.2 The effect of the curvature of space

When the movement of light is in a radial direction relative to the gravity generating source, then the slope of curved space also plays a role. There is then another factor $\sqrt{g_{tt}}$ involved, as pictured in figure 8.1. Curved space means that the light, already 'slowed down' by the slower running clock, also moves at an 'angle' to Schwarzschild space, so that the movement in the radial direction is further retarded by a factor $\sqrt{g_{tt}}$. The local *radial speed* of light transforms to the distant reference frame by a factor g_{tt}, just like any other radial velocity.

8.3.3 The time delay of light

One can say that as observed by a distant observer, the *transverse speed* of light is $\sqrt{g_{tt}} = \sqrt{1 - 2GM/r}\, c$ and the *radial speed* is $g_{tt} = (1 - 2GM/r)c$, where r is the radial Schwarzschild coordinate distance from the centre of mass M. Here the terms transverse and radial mean relative to the mass.

It must be said that scientists do not like this sort of talk, i.e. that the speed of light 'slows down' in a gravitational field. As touched on before, the reason for this is that if a *locally* stationary inertial observer performs a localized measurement of the speed of light in the gravitational field, the result is always c. It is only when a stationary distant inertial observer measures the time that light takes to pass close to a gravitational source, that there is a real (not apparent) delay.

For a light beam that grazes the surface of the Sun, say between Earth and Mars at superior conjunction,* the photon path is almost purely radial

*Superior conjunction occurs when a planet is on the opposite side of the Sun, as viewed from Earth.

relative to the Sun. The 'speed' of light can be taken as approximately $g_{tt} = 1 - 2\bar{M}/r$ for both legs of the trip (as a first approximation).

The time delay is defined as the round trip difference between 'Newtonian time', where $c = 1$ and 'relativistic time', where $c = 1 - 2\bar{M}/r$. Since r changes continuously during the trip, the time delay must be obtained by integration of time over the distance. For a photon passing relatively close to the Sun, the following approximation holds:

$$\Delta t_d \approx 240 - 20\ln(\frac{d^2}{r_p})\ \mu\text{s},$$

where d is the distance of closest approach to the Sun, expressed in solar radii, r_p the distance of the planet from the Sun in astronomical units (AU) and In is the natural logarithm. The delay for the 42 minute round trip of a Sun-grazing photon to Mars at superior conjunction and back works out to about 250 μs.

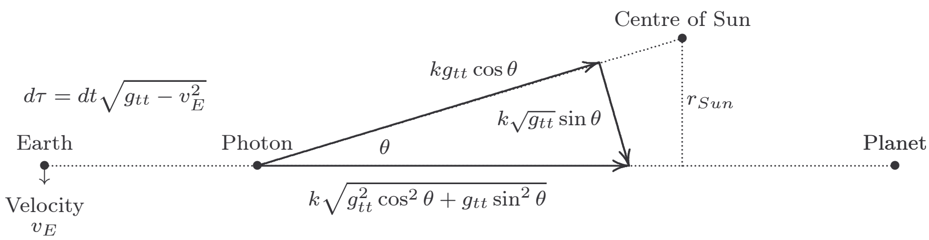

Figure 8.2: The arrowed triangle represents the radial and transverse components and the resultant vector for a Sun-grazing photon's velocity (for one position of the photon). As used before, $g_{tt} = 1 - 2\bar{M}_{Sun}/r$, where r is the radial distance between the photon and the centre of the Sun. The constant k is just an arbitrary scale factor. Earth time is represented by $d\tau$ and Schwarzschild coordinate time by dt.

When corrected for the fact that there is a transverse component involved in the path of the photon and also for the fact that an observer on the surface of the Earth is not quite a 'stationary, distant inertial observer', the predicted time delay for a round trip to Mars is around 200 μs. Figure 8.2 illustrates the effects. The 'effective speed of light' in Schwarzschild coordinates can be closely approximated by

$$c_S \cong c\sqrt{g_{tt}^2 \cos^2\theta + g_{tt}\sin^2\theta},$$

in whatever units are chosen for c. Here θ is the coordinate angle between the photon path and the radial from the mass $\bar{M}$. The speed of a photon is slightly faster in transverse directions than in radial directions and therefore decreases the predicted time delay. Further, the total time dilation at Earth's average orbital distance from the Sun (r_E) and average orbital speed (v_E), is

$$\frac{d\tau}{dt} \cong \sqrt{1 - \frac{2\bar{M}_{Sun}}{r_E} - v_E^2}\ ,$$

where $d\tau$ represents the rate of clocks on Earth and dt the rate of the distant* reference clock. The slight ellipticity of Earth's orbit is ignored.

*Here a distant clock is one at rest relative to the Sun and far enough away to be considered as measuring Schwarzschild time.

The fact that clocks on Earth run slightly slower than distant clocks causes the measured time delay to be smaller than what it would have been if Earth clocks ran at 'distant (Newtonian) rates'. The Earth's mass and rotation and the mass of the 'target' planet also play a role, but the effects are much smaller than the experimental errors and are usually ignored. Likewise, the fact that the photon's path will be deflected slightly by the Sun, contributes negligibly to the time delay.

The first predictions for the delay during superior conjunction and the actual measurements thereof were performed by Irvin Shapiro et. al. in the early 1960s. They bounced radar signals off the planets Venus and Mercury when they were close to superior conjunction. This type of time delay has since then been called the **Shapiro time delay**.

During the late 1960s and early 1970s, Shapiro bounced radar signals off Mars near superior conjunction. He later also utilized one of the Viking Mars landers, fitted with appropriate transponders. This eliminated the uncertainty of where the radar signal bounced of the Martian surface and allowed Shapiro and his team to reduce the observational errors to smaller than 0.1% of the effect they were after.

The results were in almost perfect agreement with the predictions of general relativity. An excellent description of this delicate experiment can be found in [Will(b)]. Will also offers an up to date technical web site, [Will(c)], which is well worth a visit.*

*You might want to read chapter 11 before visiting it, due to the rather abbreviated, yet technical nature of the site.

8.4 Gravity Probe B

There are two other effects of general relativity that are potentially testable in the solar system: the dragging of inertial frames and the geodetic effect. The purpose of Gravity Probe B was to test both in the vicinity of Earth. But first, what are they?

8.4.1 Geodetic Effect

The geodetic effect is caused directly by curved spacetime. It is somewhat similar to parallel movement* on curved three-dimensional surfaces.

*Relativists call it parallel transport, meaning never rotating relative to the local surface.

Consider an arrow pointing due north on Earth's equator at zero longitude. Now parallel transport it to the north pole along the zero longitude meridian. From there, parallel transport it down the 90 degrees meridian to the equator.

The arrow will now point due east, while it never rotated relative to the (two dimensional) surface. This is similar to the geodetic effect in curved four dimensional spacetime.

Gravity Probe B does a similar thing by parallel transporting a gyroscope through the curved spacetime surrounding Earth. During each orbit of the spacecraft, the spin axis of a very stable on-board gyro is predicted to tilt by a tiny amount. Do that for a year or more and the tilt becomes measurable.

8.4.2 Dragging of Inertial Frames

The dragging of inertial frames happens around a spinning mass. General relativity predicts that a spinning mass drags spacetime around itself, almost like a spinning ball in a fluid will drag the fluid around with it. Particles submerged in the fluid will then be dragged with the fluid.

Compact, spinning bodies like neutron stars and black holes are predicted to drag spacetime around by a significant amount. The Earth's rotation is predicted to drag the spacetime around by a very small amount, but over a long period it might become measurable.

This is what Gravity Probe B attempts to achieve. For a spacecraft in orbit over the poles, the frame dragging will tilt a gyro in a different direction than what the geodetic effect will do. The two effects can thus be measured with the same gyroscope. For redundancy, the science package included four identical, very precise gyros.

At the time of writing, Gravity Probe B has been up there for more than a year, of which just about one full year was a "science run". The data analysis will take another year and the results are expected early in 2007.

8.5 The spacecraft

At a 650km high orbit, the satellite completed more than 5000 orbits during the year-long science run. The expected results are: a geodetic shift of 6. 6 arc-sec per annum and a frame dragging shift 0.042 arc-sec per annum in the spin axis of the gyros.

It is clear from the expected results that the four gyros had to be of extreme accuracy - a precision of 0.0005 arcseconds was the target. How did they do it? This quote from one of NASA's websites [GP-B] says a lot:

"**April 26, 2004:** Engineers don't often indulge in poetic flourish when discussing the things they build. So when words like 'beautiful' and 'elegant' and 'artful' frequently cross the lips of scientists and engineers as they talk about the design of Gravity Probe B (GP-B), one might suspect that this spacecraft is truly something special.

"Telescopes. Gyroscopes. Superconducting lead bags and SQUIDs. These are odd materials for art. Among engineers and physicists, though, there's no doubt: Gravity Probe B is a masterpiece." Read more about this "technological *tour de force*" on NASA's website [GP-B].

It will be very interesting to see how close the actual results come to the predictions of general relativity.