Chapter 3

Linear acceleration and Relativity

when world-lines
are
not straight

So far we have only considered objects moving uniformly relative to some inertial reference frame. Such objects have straight world-lines in Minkowski spacetime. It then follows that objects undergoing acceleration should have curved world-lines. Although accelerated frames of reference are not strictly part of special relativity, we will investigate them as a link between special an general relativity. Before we go there, it will be useful to consider a different way of picturing spacetime—the so called space-propertime diagram.

3.1 Space-propertime diagrams

The usual way of drawing spacetime diagrams is to use coordinate time versus coordinate space. Some interesting insights can be gained by drawing propertime intervals against coordinate space intervals. Such diagrams are also known as Brehme diagrams.

In simple terms it means that if Jim is stationary in the inertial coordinate system and Pam is moving uniformly relative to him, we plot Jim's space intervals and Pam's propertime intervals against each other. Such a combination has been described by some authers, e.g., [Thorne], as *"my space and your time"*.

One of the consequences of using this curious mix of coordinates is that one can divide 'my space' by 'your time' and get an answer that exceeds the speed of light. Never the less, if one use it with caution, space-propertime diagrams are very useful and sometimes surpass the insight that can be gained from normal spacetime diagrams.

If Pam is also moving uniformly, we can plot Pam's space intervals and Jim's propertime intervals; it gives an equivalent, yet mirror-imaged picture, as shown in figure 3.1.

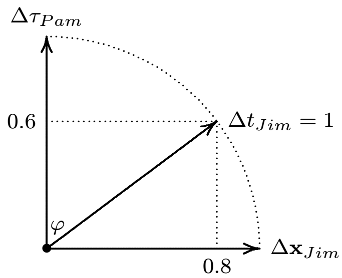


Figure 3.1: A space-propertime diagram where Pam moves uniformly at 0.8c relative to Jim (left) and where Jim moves uniformly at -0.8c relative to Pam (right). Note that the distances and times are intervals between events.

The figure is drawn for a very fast relative velocity (0.8c). What is immediately obvious, is that the slope of this coordinate time arrow (t) is less than 45 degrees (at least for positive velocities). Contrast this with Minkowski spacetime diagrams, where the slope of the (positive) speed of light is exactly 45 degrees.

What is the slope of the space-propertime arrow for light? The answer is zero or 180 degrees, depending on direction. This is so because a lightlike interval between two events represents a zero propertime interval.

The angle that the time arrow makes with the propertime axis (τ) is

$$\varphi = -\arcsin(\dot{\mathbf{x}}) = -\arcsin(v/c),$$

in accordance with the answer to the question above, where positive angles are taken as per the usual convention.

We will use the space-propertime concept extensively in the discussion of accelerated objects that follows. We will see in later chapters that gravitational acceleration is also easily visualized in space-propertime diagrams.

3.2 Accelerated frames of reference

It is obvious that if objects are accelerated and their speeds change relative to the reference frame, they must have curved space-propertime arrows. Curves are normally characterized by curvature or radius of curvature and a centre of curvature, all of which may change along the curve.

A circle is the simplest case, where the curvature is constant and equals the inverse of the radius of the circle. The centre of curvature is simply the centre of the circle. Most other curves can be broken up into a large number of circle segments, each part of the so called *circle of curvature* for a point on the curve.

A straight line is an obvious exception, because the curvature is zero, with an infinite radius of curvature. We will use the circle of curvature to construct space-propertime diagrams for accelerating frames of reference, a simple example of which is shown in figure 3.2.

This is a circle segment and the curve has constant curvature $1/R$. Constant curvature does however not represent constant acceleration—not in the rest frame, neither in the accelerating frame. The reason for this will become clear later in this section.

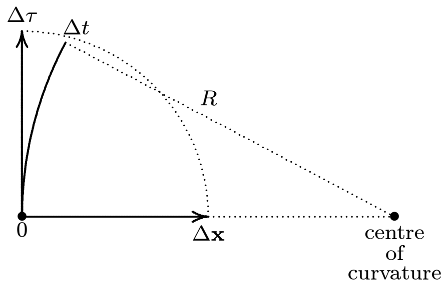


Figure 3.2: The wordline of an accelerating frame, shown here with a constant curvature around the centre of curvature and radius of curvature R. The curved worldline $(0, \Delta t)$ of the accelerating frame has the same length as the $\Delta\tau$ axis length $(0, \Delta\tau)$.

3.3 Transformation of acceleration

As a first step in studying accelerated frames, we need to know how to transform an acceleration measured inside a moving frame of reference back to the inertial reference frame. The acceleration $\ddot{\mathbf{x}}'$ measured inside a moving frame $\mathbf{x}'$ transforms to the rest frame as

$$\ddot{\mathbf{x}} = (1 - \dot{\mathbf{x}}^2)^{\frac{3}{2}}\ \ddot{\mathbf{x}}', \tag{3.1}$$

where $\dot{\mathbf{x}}$ is the instantaneous speed of the accelerating frame relative to the inertial frame of reference. This result is given as face value, but it is analyzed further in the box on page 62.

The value $(1 - \dot{\mathbf{x}}^2)^{\frac{1}{2}}$ is the velocity time dilation factor. Therefore, the rest frame measures an acceleration that is *three factors of velocity time dilation smaller* than what is measured inside the accelerating frame.

As an example, accelerate a test object at 1g (from rest) inside a spaceship that is moving uniformly at 0.6c relative to the inertial reference frame. Assume that the acceleration is in the direction of motion. An observer stationary in the *reference frame* will measure the test object's initial acceleration as

$$\ddot{\mathbf{x}} = (1 - 0.6^2)^{\frac{3}{2}} = 0.512 \text{ g}.$$

This will also hold for the case where the whole spaceship is accelerating relative to the reference frame. The velocity will then be changing continously, so the transformation only holds instantaneously.

3.4 The effect of acceleration on time

Let the spaceship accelerate uniformly at 1g as measured by some form of accelerometer on board of the spaceship. Let Pam ride the accelerating spaceship, while Jim remains stationary in the inertial reference frame.

Jim will measure Pam's acceleration as declining with her speed until, as she approaches the speed of light relative to him, he will observe her acceleration to approach zero.

In Pam's (accelerating) frame of reference, the measured acceleration will however remain at 1g for as long as her ship's propulsion system functions properly.

If Jim take Pam's constant acceleration ($\ddot{\mathbf{x}}_P$), he can transform it to his frame of reference as

$$\ddot{\mathbf{x}}_J = (1 - \dot{\mathbf{x}}^2)^{\frac{3}{2}}\ \ddot{\mathbf{x}},$$

where $\dot{\mathbf{x}}$ is Pam's changing relative speed at any moment.

Jim then integrates this expression twice with respect to time and finds the relationship between the space distance ($\mathbf{x}$) that Pam travel in his frame and the proper time (τ) on board her ship for any given constant acceleration.

In other words, Jim can plot a space-propertime diagram for Pam's ship. The result of the double integration is the exponential function*

*Relativists call this 'hyperbolic motion' and writes it as $\mathbf{x} = \frac{cosh(\ddot{\mathbf{x}}'\tau')-1}{\ddot{\mathbf{x}}'}$, (e.g., [Thorne], notes section, referred to page 37), which boils down to the same thing.

$$\mathbf{x}_J = \frac{e^{\ddot{\mathbf{x}}\tau} + e^{-\ddot{\mathbf{x}}\tau} - 2}{2\ddot{\mathbf{x}}}. \tag{3.2}$$

The function is shown graphically in Figure 3.3.

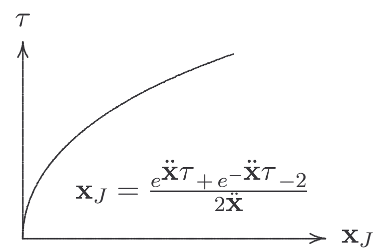


Figure 3.3: The space-propertime path for an object accelerating at a constant rate $\ddot{\mathbf{x}}$, as measured in the accelerating frame.

It is not easy to extract Pam's propertime (τ) out of the equation. Therefore it is left in the reciprocal form ($\mathbf{x}_J$ as a function of $\ddot{\mathbf{x}}$ and τ).

For a very long space trip, even with moderate acceleration, $\ddot{\mathbf{x}}\tau \gg 1$, meaning $(e^{-\ddot{\mathbf{x}}\tau} - 2)$ becomes negligibly small compared to $e^{\ddot{\mathbf{x}}\tau}$.

The equation for the space interval in Jim's frame then reduces to

$$\mathbf{x}_J \approx \frac{e^{\ddot{\mathbf{x}}\tau}}{2\ddot{\mathbf{x}}}. \tag{3.3}$$

From this equation Pam's time interval τ can be easily extracted as

$$\tau \approx \frac{ln(2\ddot{\mathbf{x}}\mathbf{x}_J)}{\ddot{\mathbf{x}}}. \tag{3.4}$$

We will later use these approximations to analyze long duration space travel under constant acceleration.

Now back to the curve of figure 3.3. It is clear that the *curvature* is at a maximum at the origin $(\mathbf{x}_J, \tau = 0)$ and at a minimum when $\mathbf{x}_J$ is large. At the origin, the *radius of curvature* can be shown to be $R_0 = 1/\ddot{\mathbf{x}}$.

At other points on the curve, the radius of curvature is enlarged by a factor $1/(1 - \dot{\mathbf{x}}_J^2)$, as discussed later in the chapter.

Of further interest is the fact that the length of the curve equals the coordinate time (Jim's time) that elapsed. This length is given by

$$t = \frac{(e^{\ddot{\mathbf{x}}\tau} - e^{-\ddot{\mathbf{x}}\tau})}{2}, \tag{3.5}$$

where t is the time that Jim measures.

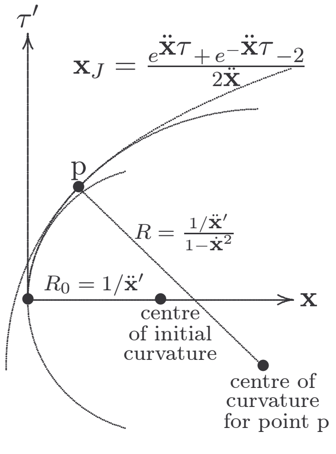


Figure 3.4: Circles of curvature along the acceleration curve can be obtained from the slope of the curve. The velocity $\dot{\mathbf{x}}$ is a function of the slope of the curve and with the velocity known, the radius of curvature R and the centre of curvature can be found geometrically. From the equation for R it is clear that as velocity $\dot{\mathbf{x}}$ tends to the speed of light, the radius of curvature R will tend to infinity and the curvature will tend to zero, meaning acceleration relative to the rest frame ($\mathbf{x}\tau'$) will tend to zero.

The centre of curvature for any point is found by drawing a line normal to the curve in the direction that the line curves and with a length equal to the radius of curvature R. In order to find the radius of curvature, we need both the velocity ($\dot{\mathbf{x}}$) and the acceleration ($\ddot{\mathbf{x}}'$) at that point of the curve.

We know the acceleration and the velocity can be found from the slope of the curve, because the slope represents the velocity vector at that point. Once we have velocity and acceleration, the radius of curvature is known and we know where the centre of curvature is located, as shown in Figure 3.4.

When we do not have a curve and want to construct it from scratch, the method is similar. The simplest algorithm involves keeping track of the angle φ through which the velocity vector has turned, as shown in figure 3.5. Decide on a time increment Δt and from the old curve position (p) and the old centre of curvature, draw a circle segment Δt, giving $\Delta\varphi = \Delta t/R$.

You now have $\varphi_{new} = \varphi + \Delta\varphi$. For this new position, find the new radius of curvature, which together with φ_{new} gives the new centre of curvature. The process can then be repeated for as many cycles as you like. By making Δt very small, a highly accurate curve can be obtained. The box on page 64 gives a programming algorithm for constructing such a curve.

The algorithm is simplified by first doing a $\Delta\varphi$ rotation with distance R on the x-axis of a 'dummy' coordinate system $\mathbf{x}, y$, obtaining $\Delta\mathbf{x}$ and Δy. Then a coordinate system rotation is performed through the angle φ, giving the space and propertime movements $\Delta\mathbf{x}$ and $\Delta\tau'$. The construction method of

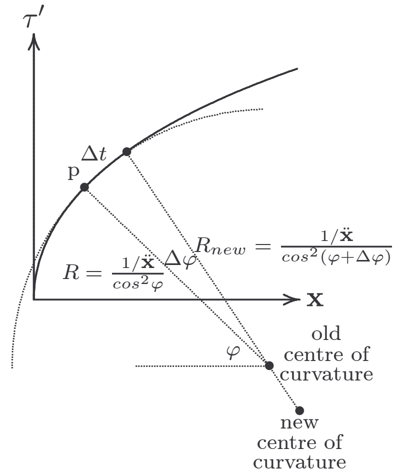


Figure 3.5: Construction of the acceleration curve in space-propertime. The coordinate time interval Δt together with the current radius of curvature R determines the new position on the curve and also $\Delta\varphi$, giving the new φ, which determines a new radius of curvature and a new centre of curvature.

finding the space-propertime curve for an accelerating frame is useful when the acceleration changes with time in a complex way. If the acceleration is kept constant, it is easy to calculate values straight from the equations.

Using the approximate formulas, we will now proceed to do a few interesting calculations.

Firstly, we will work out how long (in spaceship propertime) it will take a spaceship, accelerating at a comfortable 'one earth gravity' ($1g$), to travel a distance equal to the radius of the observable universe.

Secondly, we will determine how far the ship will go in the expected productive lifetime of a human spacefarer aboard.

If we work in geometric units of years, then 1g equals about 1 lightyear/year^2, so $\ddot{\mathbf{x}} \approx 1$. The radius of the observable universe, in light travel time, is presently 15×10^9 lightyears at most.

The proper time needed to travel a distance equal to the radius of the observable universe is

$$\tau' \;\approx\; ln(2\mathbf{x}\ddot{\mathbf{x}})/\ddot{\mathbf{x}} \;=\; ln(2 \times 15 \times 10^9) \;\approx\; 24 \text{ years,}$$

well within a productive human lifetime.

In normal reference frame time though, the journey would take as long as the present age of the universe. Assuming a near constant expansion rate, the observable universe would by then have grown to about double it's present size.

Next, let us find out how far a human spacefarer, who keeps up 1g acceleration all the time, can go in about 30 years of 'productive' propertime.

$$\mathbf{x} \;\approx\; e^{\ddot{\mathbf{x}}\tau'}/2\ddot{\mathbf{x}} \;=\; e^{30}/2 \;\approx\; 5 \times 10^{12} \text{ lightyears,}$$

about 300 times the radius of the observable universe. This means that if the universe were not expanding, our spacefarer could possibly have circumnavigated the entire universe many times!

But what is the use of space-travel at the nearly the speed of light? One will not be able to observe much! There is a space travel 'trick' for reaching a distant star system in relative comfort and then to stop there. This is so that one can observe what is going on.

The trick is to accelerate half the distance there at 1g and then reverse the engine's thrust. This way you can decelerate at 1g for the second half of the way and 'stop' near the star system. Well, approximately, at least. You should be able to observe, despite some residual velocity.

What will be the distance that you can reach if you still have say 30 years of expected life left? We shall work out how far you get in 15 years of constant acceleration and then double it, because the deceleration phase will take as long as the accelerating phase. The answer is

$$\mathbf{x} \approx 2 \times e^{15}/2 \approx 3 \text{ million lightyears,}$$

which is not all that far on a cosmological scale. The Andromeda galaxy, part of our Local Group of galaxies, is some 2 million lightyears away. If you time your mid-point carefully at around 14.5 years, you can cruise to a 'stop' somewhere inside Andromeda.

An interesting question: what maximum speed, relative to the reference frame, would you have reached at the halfway mark? The speed after a long time of sustained 1g acceleration ($\ddot{\mathbf{x}} \approx 1$) can be closely approximated by

$$\dot{\mathbf{x}} \approx 1 - \frac{1}{2\mathbf{x}^2}\,, \tag{3.6}$$

where here $\mathbf{x}$ is the coordinate distance to the halfway point (about 10^6 lightyears for the trip to the Andromeda galaxy).

The speed works out to $\dot{\mathbf{x}} \approx 0.9999999999995$. That's twelve nines after the decimal point—closer than 1 part in 10^{12} from the speed of light!

All the above calculations are in the realm of science fiction. Why? Because of (amongst other things) the following problem - where do we find a drive for the spaceship that can keep up even the modest acceleration of 1g for tens of years?

The energy required for this sort of mission cannot be carried on board the spaceship, because to get that close to the speed of light, we will have to convert just about every gram of mass of the entire spaceship into energy at 100% efficiency - which is an impossibility.

So the energy must come from some external source. How about tapping radiation coming from the stars that we pass on our way? The problem is that radiation coming from the stars behind the ship will be Doppler-shifted virtually out of existence. The same will happen to any energy that we try to beam to the ship from Earth.

Radiation from the stars ahead of the ship will be Doppler-shifted to white-hot energy, but it comes from the wrong direction. Radiation pressure on the front of the spaceship will be enormous.

There is (perhaps) one 'plausible' energy source though - the energy of the vacuum. If we can borrow some energy from the vacuum during the acceleration phase and give it back to the vacuum during the deceleration phase, we might be in business.

How such a 'vacuum drive' might work, nobody knows. If only scientists and engineers can come up with some plan! "Free energy", unfortunately, seems to be an illusion.*

*The Internet is 'flooded' with free energy schemes, some of them proposing the energy of the vacuum. So far, no credible schemes have surfaced.

Back to reality. We have seen how continuous linear acceleration slows down the clock on board a spaceship. Inevitably, the question must come up: is it the acceleration itself that does the trick, or is it the very high relative velocity that results from the continious acceleration?

The answer to this question is not an easy one. Acceleration is an absolute thing - it can be measured on board the spaceship with no reference to the outside world. Velocity is a relative thing that can only be measured with reference to the outside world.

The next section attempts to answer this question.

3.5 Desynchronization revisited

Looking at the space-propertime diagram, e.g., figure 3.4, it appears as if it is the velocity that causes the relative slowing down of the clocks. The acceleration is just the agent that creates the relative velocity.

On the other hand, looking at the approximate equation for propertime, $\tau' \approx ln(2\mathbf{x}\ddot{\mathbf{x}})/\ddot{\mathbf{x}}$, it appears as if the acceleration $\ddot{\mathbf{x}}$ is the decisive factor. But then, speed is the first time derivative of acceleration, so in a way we can use them 'interchangeably'.

By far the simplest way to comprehend the situation is to assume that linear acceleration itself has no effect on the rate of clocks and that it is the resulting relative velocity that causes relative time dilation and a desynchronization of clocks.

The following scenario illustrates the point. Assume that we have a small, incompressible laboratory floating in free space with a master clock on the floor of the laboratory.

Now set up a 'repeater' for the ground clock' against the ceiling, which we slave to the floor clock as follows: the floor clock transmits laser light pulses to the repeater, with a pulse repetition frequency (PRF) derived from the floor clock's stable frequency source.

The repeater uses these pulses to increment counters that drive the repeater display. Provided that, in setting the initial reading of the repeater, we use the normal method of subtracting the light travel time. We are therefore continously synchronizing the repeater with the floor clock.

Now accelerate the laboratory uniformly at a constant acceleration $\ddot{\mathbf{x}}$, in a direction from the floor to the ceiling. We take this as the positive $\mathbf{x}$ direction.

In the time that each light pulse is in transit from the floor to the ceiling, the ceiling picks up some extra speed due to the acceleration. The reciever is therefore moving at a higher velocity than the transmitter. One can expect a Doppler shift of the laser light frequency and also of the PRF.

For a short distance of mild acceleration, we can take the travel time of the pulse as approximately h in geometric units (e.g. metres), where h is the height of the laboratory in metres.

So, during the travel time of each laser pulse, the ceiling has picked up additional speed $\ddot{\mathbf{x}}h$. From the previous chapter, we know that this gives a Doppler shift of received PRF relative to transmitted PRF of

$$\frac{\Delta\lambda}{\lambda} = \ddot{\mathbf{x}}h,$$

which is the fractional rate at which the ceiling repeater will be 'loosing time' against the floor clock.

After a time Δt of acceleration, the ceiling repeater will have lost $\Delta t\ \ddot{\mathbf{x}}\ h$ time units. Since $\Delta t\ \ddot{\mathbf{x}} = \dot{\mathbf{x}}$, the speed change of the laboratory in the time Δt, we can say that the ceiling receiver lost $\dot{\mathbf{x}}h$ units of time, i.e., the product of the speed change and the height of the laboratory.

If we now stop the acceleration, the repeater will start to 'tick' at the same rate as the floor clock again (because there will be no more Doppler shift). The repeater will however be $\dot{\mathbf{x}}h$ units of time behind the floor clock, as viewed by the inertial reference frame in which the laboratory was stationary before the acceleration happened.

In the laboratory frame, however, everybody will insist that the repeater is still synchronized with the floor clock—after all, the whole setup was designed to keep it synchronized!

The analysis used so far was fairly loose, because of the simplification that the light travel time is equal to the height of the laboratory. This will certainly not hold for large acceleration over long distances. Further, the accelerating laboratory is not an inertial frame and the Doppler shift calculation is not strictly valid.

Amazingly enough, when a full relativistic analysis is done, we get the same answer in a rigorous way and it is valid for any velocity change and any laboratory height. Figure 3.6 illustrates this point on a space-propertime diagram.

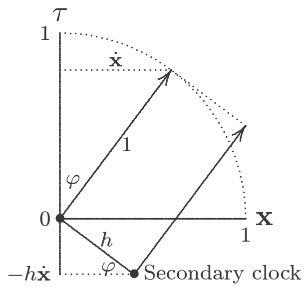


Figure 3.6: The unity space-propertime vector ν represents a clock moving at velocity $\dot{\mathbf{x}}$ relative to coordinate system $\mathbf{x}, \tau$. A secondary (synchronized) clock, stationary relative to the primary, is located at a distance (h) from the origin, as measured in the moving frame. Because $\dot{\mathbf{x}} = \sin\varphi$, the desynchronization offset (secondary - primary) $= -h\sin(\varphi) = -h\dot{\mathbf{x}}$.

Although desynchronization must be viewed as a relative velocity effect, there is a subtle second order effect caused directly by the acceleration.

Relative to the inertial reference frame, the laboratory suffers increasing Lorentz contraction as it's velocity in direction of movement increases. Relative to an inertial frame, the ceiling clock thus always travels at a lower velocity than the floor clock and suffers less time dilation.

The effect is loosely illustrated in figure 3.7. Because acceleration has an absolute nature, this difference can be measured absolutely.

An 'experiment' to test this can be done as follows: put two identical clocks on the floor of the accelerating spaceship and synchronize them. Then move one clock slowly to the ceiling, leave it there for a long time without any attempt to synchronize it to the floor clock. Finally, bring the ceiling clock slowly back to the floor again.

When the two clocks are directly compared, the clock that spent time at the ceiling of the accelerating laboratory will be ahead of the floor clock. This effect is absolute as compared to the effect of desynchronization. The latter is relative in the sense that it can only be measured over a distance, using light or other signals that travel at the speed of light.

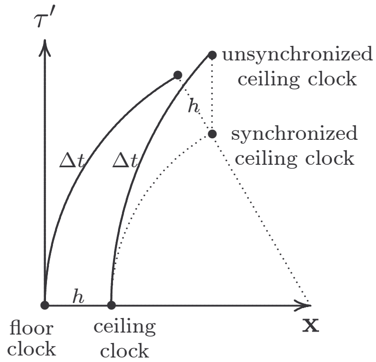


Figure 3.7: The $\mathbf{x}\tau'$ paths of the floor and ceiling clocks, separated by height h, being accelerated from rest in an inertial frame of reference, where the ceiling clock is not being synchronized to the floor clock. The ceiling clock suffers less acceleration because of accumulating length contraction due to the increasing velocity. However, the curved space-propertime pathlength of both clocks must be the same, i.e. $= \Delta t$. The only solution possible is where the unsynchronized ceiling clock's propertime is ahead of that of the floor clock, as shown. The apparent $\mathbf{x}\tau'$ position of a synchronized ceiling clock is also shown.

This effect is directly related, but not quite equivalent to *gravitational time dilation*, as will be discussed fully in a later chapter.

The described absolute effect of acceleration on the ceiling clock is much smaller than the 'apparent running fast' of the ceiling clock caused by the changing desynchronization. In just about all practical situations it can be ignored as insignificant.

This just about wraps up acceleration in a gravity free environment. In the next chapters, we will look intensively at gravitational acceleration.

3.6 Summary of Special Relativity

We have learned that the best way to view special relativity is by means of the space interval and the time interval between events. This removes (and prevents) paradoxical conclusions to be drawn. An observer that is present at two events always measures a time interval that is shorter than the time interval measured by any inertial observer not present at both events.

The Lorentz transformation is the 'tool' for transforming time and space intervals from one inertial reference frame to another. This is possible because the spacetime interval is a constant for any two events. The spacetime interval is a function of the time interval and the space interval.

If the space interval exceeds the distance that light can travel in the corresponding time interval, the spacetime interval is called 'space-like'. If the time interval exceeds the time that light will take to travel the corresponding space interval, the spacetime interval is called 'time-like'. The borderline between the two is called 'light-like' (what else?).

Momentum and energy in relativistic dynamics differ from the same concepts in Newtonian dynamics. In special relativity, the difference is coupled to the velocity time dilation factor. If any material object is moving at a speed approaching that of light in a reference frame, its momentum and energy approaches infinity in that reference frame.

In special relativity, Doppler shift of electromagnetic signals also differs from the equivalent effect in Newtonian dynamics. We have seen that the difference only shows up in one-way Doppler shifts. this is due to the 'absolute rest frame' effects of Newtonian dynamics. In two-way, round trip signals, the rest frame effects cancel out and the two theories mentioned gives the same result.

Lastly, we have examined linear acceleration in this chapter. Acceleration does not affect atomic clocks directly. However, the fact that the speed of the accelerated clock changes relative to whichever inertial reference one chooses, causes an effect on clocks.

We have seen that the one-way distance that could be accomplished in a realistic human spacefarer's lifetime is extraordinary... provided that the traveler's spaceship is linearly accelerated to very close to the speed of light.

If the ship is continously being accelerated, its average speed will be very close to the speed of light in virtually every inertial frame of reference. Hence the 'longevity' of the spacefarer as calculated in an inertial reference frame.

Having had a taste of acceleration in the gravity-free environment, it is now time to move on to general relativity, the realm of gravity and gravitational acceleration. The next seven chapters are devoted to the this very important part of physics—important because we are all experiencing the effect of gravity every moment of our lives... well just about!

Transformation of acceleration details

Set up an inertial reference frame $(\mathbf{x}, t)$ and another inertial frame $(\mathbf{x}', \tau')$, moving at a speed $\dot{\mathbf{x}}$ relative to the reference frame. Now let an observer inside the moving frame accelerate a test object with a constant acceleration $\ddot{\mathbf{x}}'$, for a time $\Delta\tau'$, starting from rest in the moving frame. If the time interval is small, the test object would have acquired a speed $\Delta\dot{\mathbf{x}}' = \ddot{\mathbf{x}}'\Delta\tau'$ relative to the moving frame. How will the reference frame $\mathbf{x}, t$ measure this acceleration?
Firstly, the additional speed $(\Delta\dot{\mathbf{x}})$, *as measured by the reference frame*, can be obtained from the law for relativistic summation of velocities, as

$$\Delta\dot{\mathbf{x}} = \frac{\dot{\mathbf{x}} + \Delta\dot{\mathbf{x}}'}{1 + \dot{\mathbf{x}}\Delta\dot{\mathbf{x}}'} - \dot{\mathbf{x}}.$$

Secondly, to find the acceleration measured by the reference frame, we also need to transform the time interval $\Delta\tau'$ to the rest frame, using the time dilation factor, obtaining

$$\Delta t = \frac{\Delta\tau'}{\sqrt{1 - \dot{\mathbf{x}}^2}}.$$

Now we can find the acceleration measured by the reference frame through $\ddot{\mathbf{x}} = \Delta\dot{\mathbf{x}}/\Delta t$. After a bit of algebraic juggling, we obtain the acceleration relative to the reference frame as

$$\ddot{\mathbf{x}} = \frac{(1 - \dot{\mathbf{x}}^2)^{\frac{3}{2}}}{1 + \dot{\mathbf{x}}\Delta\dot{\mathbf{x}}'}\ \ddot{\mathbf{x}}'.$$

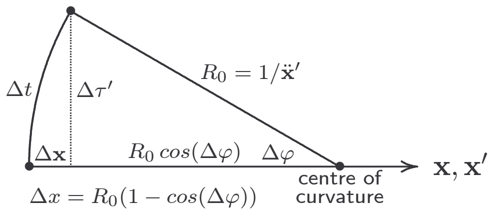
In the limit, where the time interval is so short that the change in velocity becomes negligible $(\Delta\dot{\mathbf{x}}' \to 0)$, the denominator term approaches 1 and the transformation equation approaches

$$\ddot{\mathbf{x}} = (1 - \dot{\mathbf{x}}^2)^{\frac{3}{2}}\ \ddot{\mathbf{x}}'.$$

So the acceleration transformation can be viewed as three 'velocity time dilation' transformations, i.e., $(\sqrt{1 - \dot{\mathbf{x}}^2}\,)^3$.

About acceleration and radius of curvature

The first figure below shows a circular arc Δt, defined by radius of curvature R_0, rotated through angle $\Delta\varphi$ off the $\mathbf{x}$-axis. In the limit $\Delta t, \Delta\varphi \to 0$, the results are as shown below. It is the same as the Newtonian acceleration for an object moving at the speed of light around a circle with radius R_0.



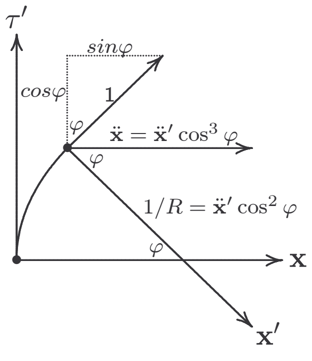
$\Delta x = R_0(1 - cos(\Delta\varphi))$

$\Delta t = R_0\ \Delta\varphi$

If $\Delta\varphi \to 0$, then $[1 - cos(\Delta\varphi)] \to (\Delta\varphi)^2/2$, and

$$\mathbf{x} = \frac{e^{\ddot{\mathbf{x}}'\tau'} + e^{-\ddot{\mathbf{x}}'\tau'} - 2}{2\ddot{\mathbf{x}}'}$$

The above is valid for acceleration from a position of rest relative to the reference frame. Once the object has picked up some speed relative to the reference frame, the velocity vector makes an angle φ with the τ'-axis and the line to the centre of curvature makes the same angle with the $\mathbf{x}$-axis. We know that $\ddot{\mathbf{x}} = \ddot{\mathbf{x}}'(\sqrt{1-\dot{\mathbf{x}}^2})^3 = \ddot{\mathbf{x}}' \cos^3\varphi$, so the radius of curvature at angle φ must be $R = 1/(\ddot{\mathbf{x}}' \cos^3\varphi) \times \cos\varphi = 1/(\ddot{\mathbf{x}}' \cos^2\varphi)$, shown below in inverse (acceleration) form.



Algorithm for programming a relativistic acceleration curve by using radius of curvature R

Understanding of the algorithm requires some programming experience, as it is written in a form of pseudo code. Text starting with " is a comment and not part of the algorithm.

Start Algorithm

Initialize the following variables as double precision floating point:

$acc = value\ of\ choice$	"acceleration $\ddot{\mathbf{x}}'$
$dt = value\ of\ choice$	"time interval Δt
$R = 1/acc$	"initial radius of curvature
$phi = 0$	"initial velocity vector rotation angle φ
$dphi = dt/R$	"incremental rotation angle $\Delta\varphi$
$\mathbf{x} = 0$	"initial space displacement
$tau = 0$	"initial propertime displacement τ'
$d\mathbf{x} = 0$	"initial space increment $\Delta\mathbf{x}$
$dtau = 0$	"initial propertime increment $\Delta\tau'$
$d\mathbf{x}a = 0$	"intermediate value $\Delta\mathbf{x}$
$dy = 0$	"intermediate value Δy

Repeat from here

Output $\mathbf{x}$ and tau to text or graphics device

$d\mathbf{x} = R \times (1 - COS(dphi))$	"$\Delta\mathbf{x}$ after rotation
$dy = R \times SIN(dphi)$	"Δy after rotation
$d\mathbf{x}a = d\mathbf{x} \times COS(phi) + dy \times SIN(phi)$	"$\Delta\mathbf{x}'$ after rotation
$dtau = dy \times COS(phi) - d\mathbf{x} \times SIN(phi)$	"$\Delta\tau'$ after rotation
$\mathbf{x} = \mathbf{x} + d\mathbf{x}a$	"new $\mathbf{x}$
$tau = tau + dtau$	"new τ
$phi = phi + dphi$	"new φ
$R = 1/(acc \times (COS(phi))^2)$	"new R
$dphi = dt/R$	"new $\Delta\varphi$

Repeat until terminating condition is satisfied

End Algorithm

With a suitably small value for Δt, this algorithm can be used to calculate the space-propertime curve for an accelerating object to any accuracy you want. Further, a new acceleration can be calculated for every cycle of the programmed loop to simulate a rocket with changing acceleration as it's fuel burns up.