Chapter 2

Special Relativity for Engineers

clocks
energy/momentum
Doppler shift

In this chapter, we will remain (more or less) inside the realm of special relativity and look at the measurement problem posed by the seemingly peculiar behavior of light, how clocks are synchronized and how energy, momentum and Doppler shift are represented in special relativity.

2.1 The peculiar nature of light

We have seen that there is apparently no inertial frame in which light (or any other electromagnetic propagation) is at rest. For every uniformly moving material object there must exist an inertial frame in which the object is at rest. In that frame of reference, space and time appears completely normal, whatever "normal" might mean.

Light does not have a "normal" reference frame and it is not wise to try and cast it into such a frame. As far as we know, the reference frame for light is singular and totally undefined. On a spacetime diagram, this becomes clearer if we draw the time axis and the space axis for a moving material object.

As we will analyse a little later, the time and space axes for an inertial observer that is moving relative to a reference frame is skewed, as shown in figure 2.1. When the speed of such an observer approaches the speed of light, the t and $\mathbf{x}$ axes tend to become one and the same. In this limit only spacetime events where $ct = \mathbf{x}$, i.e. light-like intervals, are possible.

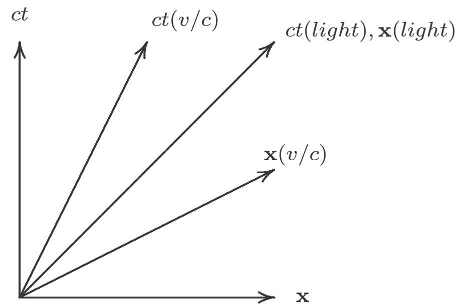

Figure 2.1: A Minkowski spacetime diagram showing the effective time and space axis of a material object moving at velocity v in a reference system $ct, \mathbf{x}$. As the velocity approaches that of light $(v \to c)$, the moving system's time and space axis almost coincide and all points approach a singular, undefined state relative to the reference frame.

We are bound to use this strange phenomenom to observe and measure most of what we are interested in. Distances, for example, are always measured directly or indirectly by using some form of electromagnetic propagation.

One can perhaps say that if we use light to measure distance, it is completely natural that we will always get the same value for the speed of light!

It is not quite that simple though. There are other factors that also have to come into play. This brings us to the question: how is the speed of light measured?

Measuring the speed of light. The most common procedure for obtaining the speed of light is to time the two-way travel of light over a known space interval. The reason for specifying a two-way experiment is that the same clock can then be used for the timing of the travel time of light.

As soon as the one-way speed of light needs to be measured, one needs two clocks—one at the starting point and one at the end point of the space interval. These two clocks need to be synchronized somehow, otherwise one have no idea of how long the light took to cover the space interval.

As we will soon see, there are more than one way in which two clocks can be synchronized—and the method chosen will influence the answer obtained for the one-way speed of light.

The Galileian (or Newtonian) synchronization of clocks. In Galileian spacetime, it is mostly a trivial exercise to synchronize remote clocks, because one can put two clocks together and synchronize them and then move one clock to a new location without upsetting the mechanism in any way. If the clocks are identical, they will stay synchronized.

If this method is not practical for some reason, one can send a messenger from the one clock to the other and, provided that we know the precise distance and the speed of the messenger, we can synchronize the clocks perfectly. The "messenger" could in principle be any material object that we shoot from the one clock's position to the other's at a known speed.

The "object" can also be light (photons). However, we will then have to know the precise velocity vector of the inertial frame relative to the aether. Why? Because in Newtonian dynamics, the velocity vector of the transmitter relative to the aether influences the speed of light.

Even if we did know the velocity of the inertial frame relative to the aether, it would be somewhat circular to synchronize clocks by means of light signals and than use those clocks to measure the speed of light.

The Einstein synchronization of clocks. The trivial synchronization method mentioned for the Galileian spacetime (moving one clock) cannot be used in Minkowski spacetime. Any relative movement of the clocks may desynchronize the clocks, at least to some degree.

We can however use a "messenger" with a constant known speed, because that is equally valid in Minkowski spacetime. Einstein used his L-principle to postulate that we can use light to synchronize any two clocks that are stationary in an inertial system.

However, light is just like a particle traveling at constant speed in all directions in an inertial frame. In this case, the measurement would suffer from the same circularity as mentioned above (using light to synchronize clocks for the purpose of measuring the speed of light).

Thought experiment to measure the one-way speed of light. Somewhere in free space, construct a large symmetrical (orthogonal) cross with arm lengths $2L$ from end to end. Let the construction move inertially and without rotation relative to the distant stars. Equip the cross with four identical atomic clocks (p,q,r,s), one at each end of an arm, as shown in figure 2.2.

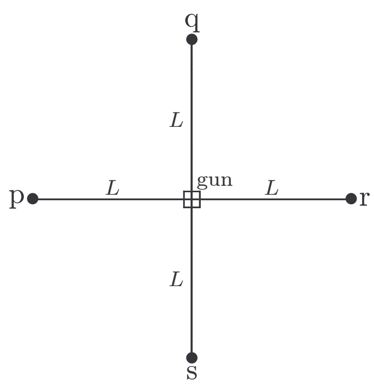

Figure 2.2: Four clocks (p,q,r,s) are situated at the ends of a perfect cross formed by bars of length $2L$. The gun at the centre can shoot single particles simultaneously and with equal speeds along all four bars.

Place a special gun in the exact centre of the cross in such a way that it can shoot material particles simultaneously at a known speed towards each of the four clocks. Now fire the gun at an arbitrary time and set each clock to a predetermined time (say zero) when the particles are detected at the corresponding clock.

If we ignore quantum uncertainty (Heisenberg's uncertainty principle), the clocks are then synchronized, because the particles will arrive simultaneously at the four clocks, at least in the inertial frame in which the cross is at rest.

Now the four synchronized clocks can be used to measure the one-way speed of light between any two of them. Let the inertial frame move at a constant speed v relative to the aether in the direction a-c.

According to *Galileian mechanics*, the one-way speed of light will depend on the direction of the light path for the four orthogonal directions as follows

$$\begin{aligned} c_{pr} &= c - v \\ c_{rp} &= c + v \\ c_{qs} &= \sqrt{c^2 - v^2} \\ c_{sq} &= \sqrt{c^2 - v^2}. \end{aligned}$$

The first two are self explanatory, while the latter two come from simple Pythagorean geometry, as the reader can easily verify.

In Einstein's special relativity, the measured speed of light will be the same for all four directions and is simply equal to c. Since there is no aether in this theory, the speed of the inertial frame relative to the aether (if it exists) is zero at all times ($v = 0$).

So how do we know that Einstein had it right? Well, the thought experiment above is for all scientific purposes equivalent to the Michelson-Morley aether-drift experiment of the early 1900s.

Although Michelson and Morley did not measure the speed of light, they have shown unequivocally that there is no significant difference in the speed of light in directions normal to and directions parallel to the movement of the Earth through the hypothetical aether.

At that stage, the only known movement of the Earth was the orbital speed of the Earth around the Sun, some 30 km/s. The small fluctuations that Michelson and Morley measured were much smaller than what was expected for such a speed. Many other experiments with improved accuracy followed, but no one was ever able to detect the expected aether-drift of Earth.

Many sceptics questioned (and are still questioning up to this day) whether a two-way experiment necessarily proves Einstein right. Some sceptics insist that a specific experiment to verify the one-way speed of light be performed in a moving inertial frame.

So why has it not been done? Well, scientists are so confident that Einstein had it right that they design particle accelerators and systems like the GPS by using the details of Einstein's relativity theory. Some scientists say that every time someone uses a GPS receiver on Earth, it is a test of the one-way speed of light in a moving inertial frame.

Perhaps not quite, but if Einstein had it wrong, you can bet your bottom dollar that linear particle accelerators and the GPS system would not have worked as advertised.

So the answer as to why a one-way test has not been done—it would be money wasted to re-establish an already well-known experimental fact. The money is more wisely spent on things that utilize that known fact!

And what if the speed of a frame changes? Firstly, while there is any acceleration, the frame is not inertial. The moment the acceleration stops, the frame will again move at a constant speed and will be inertial. How would Galileo (or Newton) have viewed the one-way speed of light in this new inertial frame?

Say we know that the frame is moving at a speed v in the direction a-c. If the change in that speed was Δv, the frame would now move at $v + \Delta v$ relative to the aether. The Galileian speed of light for the four orthogonal directions would then be

$$\begin{aligned} c_{pr} &= c - (v + \Delta v) \\ c_{rp} &= c + (v + \Delta v) \\ c_{qs} &= \sqrt{c^2 - (v + \Delta v)^2} \\ c_{sq} &= \sqrt{c^2 - (v + \Delta v)^2}. \end{aligned}$$

In this (Newtonian) view, the four clocks would still be synchronized after the acceleration, in the sense that they would at all times measure "universal time" and therefore the measured speed of light should come out as in the last set of equations.

There are immediately some objections that one can raise about this view and this set of equations. Firstly, even not allowing $v + \Delta v$ to become greater than c, the values of c_{pr}, c_{qs} and c_{sq} can become zero. This means that light cannot always propagate in those directions.

Secondly, the restriction $v + \Delta v < c$ also implies that $v + \Delta v + v_p < c$, where v_p is the speed of a particle relative to the gun. Therefore one cannot always check the synchronization of those clocks by means of particles shot from the centre, as used before.

Thirdly, even if we allow $v + \Delta v + v_p > c$ so that a particle can reach the clocks, they would no longer seem to be synchronized in the sense that they read the same time when the four particles arrive.

That is unless $v + \Delta v + v_p \to \infty$, which seems absurd. In short, the concept of simultaneity becomes problematic. Lastly, experimental results do not support the Galileian view.

In Einstein's view the measured speed of light would still be simply c in all directions relative to the frame. How could this be? Well, we would agree that the synchronization of the clocks could still be done by means of the particle gun, as before.

If the gun shot photons instead of massive particles, one can call on the wave-particle duality and argue as follows: the particle view would make the photon synchronization method equivalent to (say) an electron synchronization method. All that is required is that the speed of the specific particles are the same and remain constant during the synchronization procedure.

If one could use photons to synchronize the clocks, then the speed of light is guaranteed to come out as c during any subsequent measurement thereof! Various forms of experimental results seem to support Einstein's view so far.

This brings us to the very interesting question: would the four clocks that were synchronized before the speed change still be synchronized after the speed change Δv? The answer is no.

It has been proven (and experimentally confirmed) that acceleration *per se* does not affect the time keeping of well built atomic clocks in any measurable way. However, inertial frames in relative motion will each have their own definition of simultaneity. Einstein called this the *relativity of simultaneity*.

It can be illustrated on a Minkowski spacetime diagram as in figure 2.3. The t' axis is given by the equation $\mathbf{x} = (\Delta v/c)(ct) = \Delta vt$ and the $\mathbf{x}'$ axis by $ct = \Delta v\mathbf{x}/c$, found by setting $t' = 0$ and $\mathbf{x}' = 0$ in the appropriate Lorentz transformation equation.

For a relative speed of Δv, this gives a synchronization offset over a length L of $\Delta vL/c$ metres, or $\Delta vL/c^2$ seconds. In the $t', \mathbf{x}'$ inertial frame, any line parallel to the $\mathbf{x}'$ axis (like the dotted line $p' \rightarrow r'$) is a line of constant time in that frame, also called a line of simultaneity [Faber].

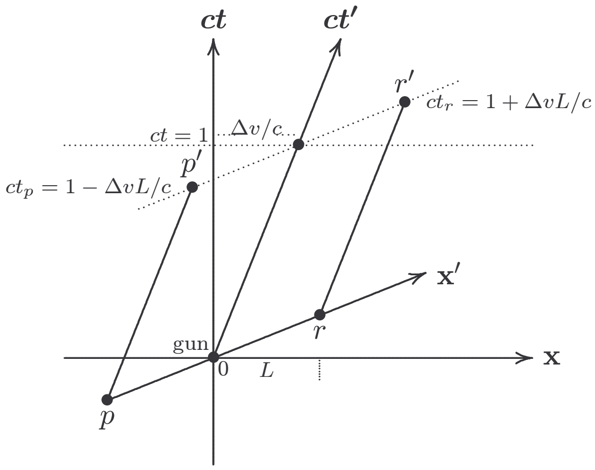

Figure 2.3: A Minkowski spacetime diagram where the primed frame is moving at a speed $\Delta v = 0.4c$ relative to the reference frame. The length of the arms are $L = 0.5$ units.

A good question at this stage: where does the synchronization offset come from? The easiest way to understand it is to consider the following.

While the frame is moving inertially, fire two particles within a short time T

in the direction of clock r. Fire the particles as close to the speed of light as available energy allows, so that the initial speed of the particles relative to the frame will approach c. *

*The particles must not be photons, because in this context, we want to synchronize the clocks so that we can measure the speed of photons.

Immediately after the second particle is fired, accelerate the frame at a moderate rate in the direction of particle movement until just before the first particle arrives at clock r.

Say the frame's speed changed by Δv, so that the speed of the particles relative to the frame will now be $c - \Delta v$. The change in time separation between the two particles passing clock r will be (in seconds per original particle period T)

$$\Delta T = \frac{\Delta v}{c - \Delta v} T \cong \frac{\Delta v}{c} T \quad \text{(for } \Delta v \ll c\text{)}$$

Convert this to the period change per particle travel time ($t \cong L/c$, which is how long the acceleration lasted), obtaining

$$\Delta t \cong \Delta T \frac{t}{T} \cong \frac{\Delta v}{c^2} L$$

seconds per particle travel time, or

$$c\Delta t \cong \frac{\Delta v}{c} L \tag{2.1}$$

metres per particle travel time. This corresponds to the synchronization offset as used in the Minkowski spacetime diagram. This result is not exact in Newton mechanics, although very close due to the low acceleration and low relative speed assumptions. As indicated by the agreement with the Minkowski spacetime diagram (based on the Lorentz transformation), the result is exact in special relativity.

There is one more interesting point to be cleared up. In which direction must the two clocks be adjusted after the acceleration? The rear clock (p) must clearly be advanced by the synchronization offset and the front clock (r) set back by the same amount.

In the Minkowski spacetime diagrams (figures 2.3 and 2.4), clock p is shown to be behind clock r, because the diagrams portray the view before the synchronization adjustments were made to the moving clocks—it is the view of the reference frame. It is only in the view of the moving frame that the new definition of simultaneity is valid.

Figure 2.4 shows the paths of the two particles as lines $0 \to p'$ and $0 \to r'$.

The particles are moving isotropically relative to the moving frame, at v'_p in both directions. In the reference frame, they do not appear to move isotropically. This is expected, because the particles have the forward motion of the gun added to the firing velocity.

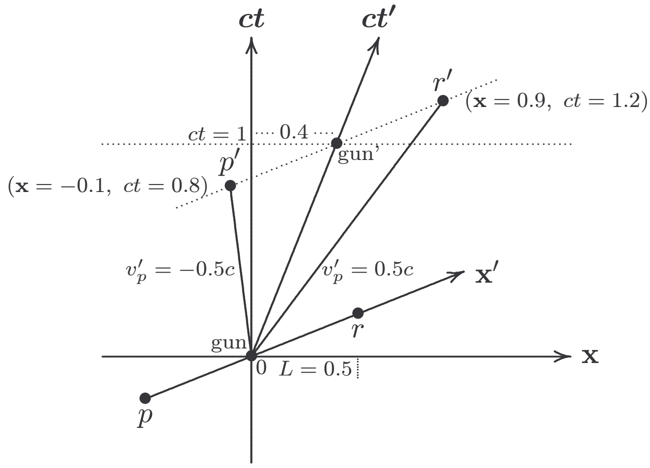

Figure 2.4: The paths of two particles moving at $v'_p \pm 0.5c$ relative to the gun respectively. The primed frame moves at $\Delta v = 0.4c$ and $L = 0.5$ units, as before.

From figure 2.4, we can see that the forward moving particle's speed in the reference frame is $v_{p+} = 0.9c/1.2 = 0.75c$ and the backward moving particle's speed is $v_{p-} = -0.1c/0.8 = -0.125c$ in the reference frame.

We have indirectly constructed the rule for *relativistic addition of velocities*:

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}. \tag{2.2}$$

If we plug in $v_1 = 0.4c$ and $v_2 = 0.5c$, we have

$$v_+ = \frac{0.4 + 0.5}{1 + 0.4 \times 0.5}c = 0.9c/1.2 = 0.75c$$

and

$$v_- = \frac{0.4 - 0.5}{1 + 0.4 \times -0.5}c = -0.1c/0.8 = -0.125c,$$

in agreement with the Minkowski spacetime diagram analysis.

This does not agree with Galileian addition of velocities, where the forward particle would have a speed of 0.4c+0.5c = 0.9c, and the backward particle would have a speed of 0.4c-0.5c = -0.1c.

Now with the synchronization of the clocks established, we can investigate how the two frames will measure the speed of light. Let one frame fire an omni directional pulse of photons when the origins of the two inertial frames coincide.

It is immaterial which frame does the firing, unlike for the case of material particles, where it does matter. Both frames will observe the photons to move isotropically in both directions and hence both will measure a photon velocity c, as indicated in figure 2.5.

Obviously, to measure the one-way speed of light, each reference frame must have its own observers posted at appropriate positions. This is so that they can read their synchronized clocks at the respective locations.

In the $\mathbf{x}', t'$ frame, the observers can just ride with clocks p and r respectively. In the reference frame x, t, two observers, with clocks synchronized in the reference frame must be positioned at a distance of plus and minus L light-seconds from the origin.

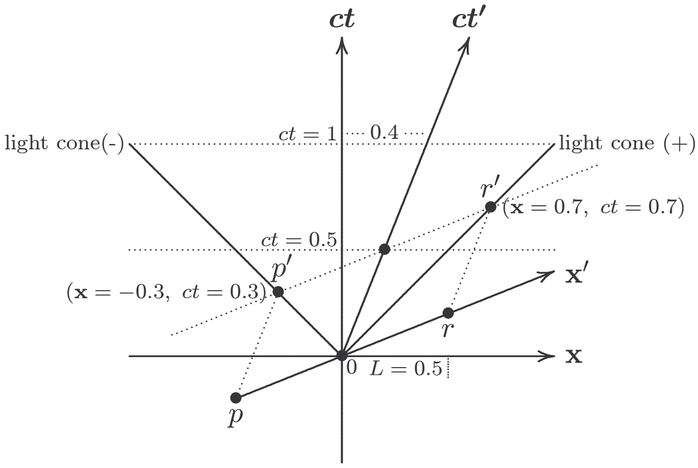

Figure 2.5: The light cone and the points of interception (p' and r') of a light pulse originating at the origin, by the two clocks p and r. Note how the same light pulse spreads isotropically in both directions for both reference frames.

One can check the isotropy of light by means of the relativistic addition of velocities rule:

$$v_{light+} = \frac{0.4 + 1}{1 + 0.4 \times 1} c = c$$

and

$$v_{light-} = \frac{0.4 - 1}{1 + 0.4 \times -1} c = -c,$$

as is expected from the Minkowski spacetime diagram.

Note that nowhere in this analysis time dilation and/or Lorentz contraction were mentioned or brought into consideration. These concepts were neatly "brushed under the carpet" by the Lorentz transformation used in determining the synchronization offset.*

*Lorentz transformations are essentially done in the equations for the t' and the $\mathbf{x}'$ axes.

If the grid lines of the moving reference frame are drawn, time dilation and length contraction become visible, as shown in figure 2.6. The horizontally and vertically measured distances between the oblique gridlines are obviously shorter than those in the reference frame. The relative lengths correspond to the time dilation and length contraction of the Lorentz transformation.

But how do we know the separations of the grid lines of the moving frame? The t' axis is "calibrated" by using the invariant timelike interval to determine the point $t' = t$ on the t' axis. Likewise, the $\mathbf{x}'$ axis is "calibrated" by using the invariant spacelike interval to determine the point $\mathbf{x}' = \mathbf{x}$ on the $\mathbf{x}'$ axis.

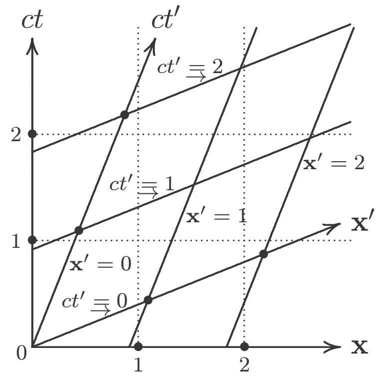

Figure 2.6: Although the oblique grid line distances of the moving coordinate system are longer than in the reference frame, the horizontal and vertical distances are shorter than the corresponding reference frame distances. This illustrates time dilation and Lorentz contraction on a Minkowski spacetime diagram.

It can be visualized by plotting the curves of the intervals on a spacetime diagram, as shown in figure 2.7. The curves are the hyperbolas

$(ct)^2 - \mathbf{x}^2 = \Delta s^2$ where $\Delta s = 2, 1, 1, 2,$ timelike and spacelike, as required.

Since $(ct')^2 - (\mathbf{x}')^2$ also equals Δs^2, an invariant quantity, the scale of the

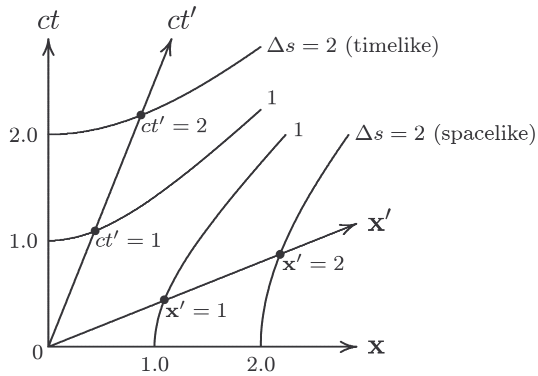

Figure 2.7: By making either t' or $\mathbf{x}'$ zero in the equations for the spacetime interval, the intersections of the hyperbolas with the moving time and space axes can be determined. This gives the "calibration" of the moving coordinate system as observed in the reference coordinate system.

t' axis is found by making $\mathbf{x}' = 0$ in the timelike equation (with $ct' > \mathbf{x}'$). This give the point where the curve crosses the t' axis. Likewise, the scale of the $\mathbf{x}'$ axis is found by making $t' = 0$ in the spacelike equation (with $\Delta\mathbf{x}' > ct'$).

Time dilation and length contraction are not "real" or absolute in special relativity, i.e., in the absence of gravity. In a way, time dilation and Lorentz contraction are just spacetime projections of one inertial frame onto another.

Absolute time and space of Galileo and Newton were replaced by the spacetime interval, which is absolute and the same in all inertial frames. This is the essence of Einstein's special relativity. And as far as we can tell so far, Einstein was right.

2.2 Mass, energy and momentum

In special relativity, mass, energy and momentum are also transformed between two inertial frames in relative motion. The meaning of Einstein's famous equation $E = mc^2$ is that the *rest energy* equals the *rest mass* of an object, where the c^2 is just a conversion constant between the conventional (SI) units for mass and for energy.

If the body moves relative to the reference frame, then it's energy increases and it can be thought to have a *moving mass* that is larger than it's rest mass. We will use the symbol ϵ for moving energy (or moving mass), to distinguish it from the Newtonian energy E.

In conventional Newton dynamics, the momentum of a moving object is $p = mv$, where v is the velocity of the mass m relative to the reference frame. Newton dynamics also defines the kinetic energy of a moving mass as $E = \frac{1}{2}mv^2$.

In Einstein's special relativity, the momentum of an object moving with velocity v relative to the reference frame is equivalent to the Newton momentum divided by the time dilation factor $\sqrt{1 - v^2/c^2}$:

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}. \tag{2.3}$$

The total energy ϵ is the vector sum of the rest energy and the relativistic kinetic energy, or

$$\epsilon = \sqrt{(mc^2)^2 + p^2c^2}, \tag{2.4}$$

shown graphically in figure 2.8. Note how the value of Newton momentum (mv) graphically relates to the relativistic momentum and energy. By solving the triangle, we get

$$\epsilon = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \tag{2.5}$$

showing that the total energy of a moving mass is the rest energy divided by the time dilation factor due to velocity.

The "strange" quantity mvc in figure 2.8 comes from multiplying rest energy mc^2 by the value v/c, keeping the units of the horizontal and vertical axes the same. The horizontal axis can be viewed as the relativistic kinetic energy. The value mvc scales linearly with the speed, from zero to mc^2 and thus rotates the total energy vector from the vertical to the horizontal.

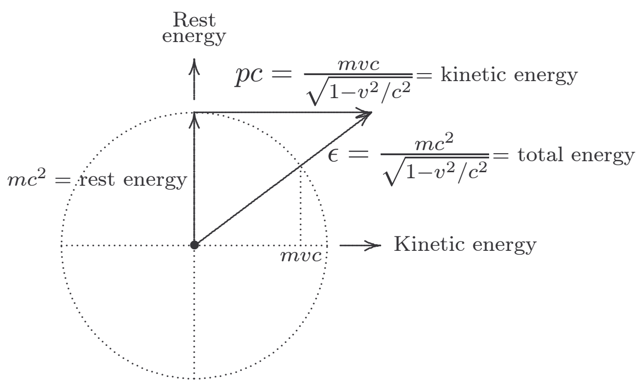

Figure 2.8: Rest energy (mc^2), kinetic energy (pc) and total energy (ϵ) of a body moving with velocity v. When $v \to c$, $mvc \to mc^2$ which is the radius of the circle. The momentum vector and the total energy vector then tend towards being parallel to each other and will meet at a very large momentum. Both momentum and total energy tend to infinity as velocity approaches the speed of light.

If $v \ll c$ the energy equation can be approximated by

$$\epsilon = mc^2 + \frac{1}{2}mv^2, \tag{2.6}$$

stressing the fact that for low velocities, Einstein's energy equation is equivalent to adding the rest energy (mc^2) to the Newton kinetic energy ($\frac{1}{2}mv^2$).*

* If some value $a \ll 1$, then $1/\sqrt{1-a^2} \approx 1/(1-a/2) \approx 1 + a/2$.

This simple relationship breaks down when velocity becomes a significant portion of the speed of light and ϵ diverges to infinity as v approaches the speed of light. The only correct interpretation is that total energy equals the vector sum of the rest mass and the relativistic kinetic energy.

When we attempt to accelerate an object to a velocity close to the speed of light, the amount of energy required is enormous. It is as if the high speed object converts most of the energy applied to it into mass instead of into extra velocity.

However, suppose we could set up an inertial frame with the same velocity as the object, so that the object is effectively at rest in the frame. If someone in this frame would measure the mass of the object, perhaps by accelerating it slightly, the mass of the object would turn out to be just it's rest mass.

Rest mass is an absolute value in the sense that all frames of reference would get the same measured mass for the same object, provided it is at rest in that frame.* If the mass is moving, the measured mass depends on how fast the object is moving relative to the frame that measures it.

*Strictly, for rest mass to be constant, the object must remain at constant temperature and pressure as well.

This is contrary to Newtonian dynamics. There the measured mass of an object remains constant, irrespective of how fast it moves relative to the frame that measures it.

In Newton's view, if an observer riding with the object measures some acceleration (a) caused by some force (F) acting on the object, then an observer in some other inertial frame,in relative motion to the first one, can do the same measurements and obtain the same measured acceleration.

This is so because, in Newton mechanics, all inertial frames will measure the same change in velocity and thus the same change in momentum in the same time interval. Velocity and momentum simply adds up in a linear fashion for all (Newton) inertial frames:

$$\dot{\mathbf{x}}_2 = \dot{\mathbf{x}}_1 + \Delta\dot{\mathbf{x}}$$

and

$$p_2 = p_1 + \Delta p.$$

In relativity, velocity and momentum do not add up linearly. It can be shown that velocity adds up as

$$v_2 = \frac{v_1 + \Delta v}{1 + v_1 \Delta v / c^2}, \tag{2.7}$$

the 'relativistic addition of velocities rule' that we met before. Momentum adds up as

$$p_2 = \frac{p_1 + \Delta p}{\sqrt{1 - v_1^2/c^2}\ \sqrt{1 - \Delta v^2/c^2}}. \tag{2.8}$$

where v_1, v_2, p_1 and p_2 are as measured in the inertial *reference frame*, while Δv and Δp are measured in an inertial frame moving at velocity v_1 relative to the reference frame.

It is equivalent to a mother ship that cruises inertially (the reference frame) and sends off a probe that reaches a stable velocity v_1 relative to the ship, so that the probe is the moving inertial frame. Then the probe shoots out a projectile with velocity Δv relative to the itself (the probe). The equations tell us how the mother ship will measure the velocity and momentum of the projectile.

The relativistic momentum summation is equivalent to the Newton summation divided by the product of two time dilation factors, one due to the base velocity and one due to the Δvelocity.

Once the new momentum is known, the new velocity can be extracted from the relationship $p_2 = mv_2/\sqrt{1 - v_2^2/c^2}$, yielding the velocity addition rule given above.

It is clear from the equations that in the extremes, momentum can tend to infinity, while velocity can never exceed 1. As a check, put $v_1 = c$ and $\Delta v = c$ into both the velocity addition and the momentum addition rules.

2.3 One way Doppler shift

To conclude this chapter on special relativity essentials, we will briefly examine the relativistic Doppler shift, as measured between inertial frames in relative motion.

First we will look at the one way Doppler shift. We start with the Newton Doppler shift for the general case where both the transmitter (Tx) and the receiver (Rx) are moving. By 'moving' we mean moving relative to Newton's "absolute frame of rest"—the frame where the speed of light equals c.

In such a case, one is forced to perform a linear transformation from the moving transmitter frame to the absolute frame and again from the absolute frame to the moving receiver frame.

In this section, the normalized speed parameter $\dot{\mathbf{x}} = v/c$ will be used for clarity. Let the transmitter Tx move at a speed $\dot{\mathbf{x}}_{\mathsf{Tx}}$ and the receiver Rx at a speed $\dot{\mathbf{x}}_{\mathsf{Rx}}$, both relative to the absolute frame.

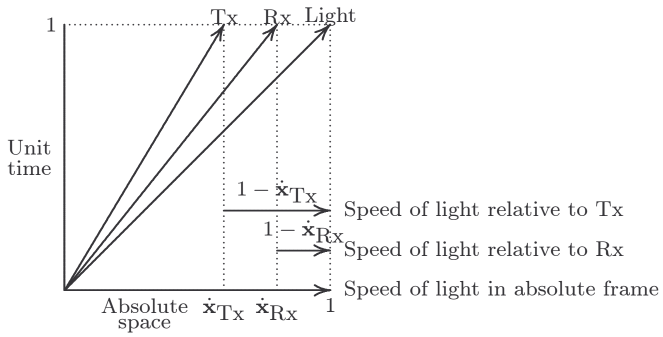

Figure 2.9: In Newton dynamics the speed of light changes relative to the moving transmitter (Tx) and the moving receiver (Rx). This causes the wavelength received by Rx to be a factor $\frac{1-\dot{\mathbf{x}}_{\mathsf{Tx}}}{1-\dot{\mathbf{x}}_{\mathsf{Rx}}}$ times the wavelength transmitted by Tx. The diagram was drawn for $\dot{\mathbf{x}}_{\mathsf{Tx}} = 0.6$ and $\dot{\mathbf{x}}_{\mathsf{Rx}} = 0.8$, giving $\lambda_{\mathsf{Rx}}/\lambda_{\mathsf{Tx}} = 2$ or $\Delta\lambda/\lambda_{\mathsf{Tx}} = 1$.

When the transmitter wavelength (λ_{Tx}) is transferred to the absolute frame, the wavelength becomes

$$\lambda_a = \lambda_{\mathsf{Tx}}(1 - \dot{\mathbf{x}}_{\mathsf{Tx}}), \tag{2.9}$$

where $1 - \dot{\mathbf{x}}_{\mathsf{Tx}}$ is the Newton speed of light relative to the Tx frame (see figure 2.9).

As a check, consider what would happen if the transmitter could move at near the speed of light relative to the absolute frame—the wavelength transmitted to absolute space would approach zero, i.e., approaching infinite frequency.

When the wavelength in the absolute frame is transferred to the Rx frame, the wavelength becomes

$$\lambda_{\mathsf{Rx}} = \lambda_a/(1 - \dot{\mathbf{x}}_{\mathsf{Rx}}), \tag{2.10}$$

where $1 - \dot{\mathbf{x}}_{\mathsf{Rx}}$ is the speed of light relative the Rx frame. Here, if the receiver could move at the speed of light relative to absolute space, the received wavelength would be infinite, i.e., no signal would be received.

From these values, the ratio of received to transmitted signal wavelength can be found as

$$\frac{\lambda_{\mathsf{Rx}}}{\lambda_{\mathsf{Tx}}} = \frac{1 - \dot{\mathbf{x}}_{\mathsf{Tx}}}{1 - \dot{\mathbf{x}}_{\mathsf{Rx}}}. \tag{2.11}$$

The *Doppler shift* is defined as the change in wavelength as a fraction of the transmitted wavelength,

$$\Delta\lambda/\lambda_{\mathsf{Tx}} = (\lambda_{\mathsf{Rx}} - \lambda_{\mathsf{Tx}})/\lambda_{\mathsf{Tx}}, \tag{2.12}$$

which works out to be

$$\frac{\Delta\lambda}{\lambda_{\mathsf{Tx}}} = \frac{1 - \dot{\mathbf{x}}_{\mathsf{Tx}}}{1 - \dot{\mathbf{x}}_{\mathsf{Rx}}} - 1. \tag{2.13}$$

The reason for having labored this point is that one-way Newton Doppler shift does not depend on the relative speed between transmitter and receiver, *but rather on their speeds relative to the absolute frame of reference*, measured in the direction of the line of sight (i.e. radial speeds).

It is only at very low speed that the usual approximation $\Delta\lambda/\lambda_{\mathsf{Tx}} = \dot{\mathbf{x}}$ holds, where $\dot{\mathbf{x}}$ is the relative radial speed. For high speeds, how would one extract the relative speed between receiver and transmitter $(\dot{\mathbf{x}}_{\mathsf{Rx}} - \dot{\mathbf{x}}_{\mathsf{Tx}})$ from the measured Doppler shift?

It is only possible if you know at least one of the speeds relative to the absolute frame as well, because there are many combinations of $\dot{\mathbf{x}}_{\mathsf{Tx}}$ and $\dot{\mathbf{x}}_{\mathsf{Rx}}$ that will give the same Newton Doppler shift. For example, speeds of 0.6c and 0.8c will give the Doppler shift

$$\frac{\Delta\lambda}{\lambda_{\mathsf{Tx}}} = \frac{1 - 0.6}{1 - 0.8} - 1 = 1,$$

which is the same as what you would get for speeds of 0.96c and 0.98c, i.e.,

$$\frac{\Delta\lambda}{\lambda_{\mathsf{Tx}}} = \frac{1 - 0.96}{1 - 0.98} - 1 = 1,$$

while the relative speeds differ by an order of magnitude (0.2c versus 0.02c). This is a consequence of the involvement of the absolute frame of reference where light has a constant speed.

In special relativity, there is no absolute reference frame, at least as far as observables are concerned, so one simply choose either transmitter or receiver as the reference frame.

Let us choose the transmitter as the 'stationary' reference frame and let the receiver move away from it at a radial speed of $\dot{\mathbf{x}}$. The received wavelength would be like the Newton receiver moving through absolute space

$$\lambda'_{\mathsf{Rx}} = \lambda_a/(1 - \dot{\mathbf{x}}_{\mathsf{Rx}}).$$

Since the receiver's distance measurements (and thus also wavelength measurements) are Lorentz contracted by $\sqrt{1-\dot{\mathbf{x}}^2}$, the received wavelength will be

$$\lambda_{\mathsf{Rx}} = \frac{\sqrt{1-\dot{\mathbf{x}}^2}}{1-\dot{\mathbf{x}}}\,\lambda_{\mathsf{Tx}} = \sqrt{\frac{1+\dot{\mathbf{x}}}{1-\dot{\mathbf{x}}}}\,\lambda_{\mathsf{Tx}}.$$

If we choose the receiver as the 'stationary' reference frame, then the transmitter is moving at $-\dot{\mathbf{x}}$ relative to the receiver. Again it is like the Newton case of transmitter transferring signal to the absolute rest frame (with negative velocity), i.e.,

$$\lambda'_{\mathsf{Rx}} = \lambda_a[1-(-\dot{\mathbf{x}}_{\mathsf{Rx}})].$$

Since the moving transmitter's distance measurements (and thus also the wavelength transmitted) are Lorentz contracted, the received wavelength will be

$$\lambda_{\mathsf{Rx}} = \frac{1+\dot{\mathbf{x}}}{\sqrt{1-\dot{\mathbf{x}}^2}}\,\lambda_{\mathsf{Tx}} = \sqrt{\frac{1+\dot{\mathbf{x}}}{1-\dot{\mathbf{x}}}}\,\lambda_{\mathsf{Tx}},$$

which boils down to the same value as for the stationary transmitter. The usual Doppler shift expression is then

$$\frac{\Delta\lambda}{\lambda_{\mathsf{Tx}}} = \sqrt{\frac{1+\dot{\mathbf{x}}}{1-\dot{\mathbf{x}}}} - 1. \qquad (2.14)$$

It can be easily shown* that when $\dot{\mathbf{x}} \ll 1$, this expression reduces to the

*By noting that for $\dot{\mathbf{x}} \ll 1$, $1/(1-\dot{\mathbf{x}}) \approx 1+\dot{\mathbf{x}}$.

Newtonian approximation $\frac{\Delta\lambda}{\lambda_{\mathsf{Tx}}} \approx \dot{\mathbf{x}}$.

This then, is the case for one way Doppler shifts. What happens if we consider the two way Doppler shifts, as applicable to Doppler radar?

2.4 Two way Doppler shift

This little section is specially for radar engineers, who might have got a little anxious, reading about the 'errors' they are making with their usual Doppler radar equations.

Here is some consolation: when a two way situation is considered in a Newton absolute frame, the effect of the absolute frame of reference cancels out!

One can laboriously go through a process of multiple translations to the absolute frame and back (four translations in all), just to find that the absolute frame disappears and you get a total wavelength ratio of

$$\frac{\lambda_{\mathsf{Rx}}}{\lambda_{\mathsf{Tx}}} = \frac{1+\dot{\mathbf{x}}}{1-\dot{\mathbf{x}}}\,,$$

or a Doppler shift of

$$\frac{\Delta\lambda}{\lambda_{\mathsf{Tx}}} = \frac{1+\dot{\mathbf{x}}}{1-\dot{\mathbf{x}}} - 1. \qquad (2.15)$$

This is precisely the relativistic two way Doppler shift, because there the wavelength is changed twice by the factor $\sqrt{\frac{1+\dot{\mathbf{x}}}{1-\dot{\mathbf{x}}}}$ (once each way). When $\dot{\mathbf{x}} \ll 1$, or in conventional units, $v \ll c$, the usual approximation yields

$$\frac{\Delta\lambda}{\lambda_{\mathsf{Tx}}} \approx (1+\dot{\mathbf{x}})^2 - 1 \approx 2\dot{\mathbf{x}}, \tag{2.16}$$

because $\dot{\mathbf{x}}^2$ is small enough to be neglected. This is the relationship that every radar engineer knows. For just about every radar measurement, even in spaceflight, this approximation is good enough.

Earth's orbital velocity is some 30 km/s. Then there are some meteorites that comes head-on towards Earth at near solar escape velocity, about 40 km/s relative to the Sun. This gives a velocity relative to Earth of near 70 km/s, or $2 \times 10^{-4}c$.

So $\dot{\mathbf{x}}^2 = 4 \times 10^{-8}$, which is still small enough for the errors to be small compared to the accuracy of current Doppler radars. It is unlikely that we will use a Doppler radar to measure anything much faster than that.

It is different in astronomy, where we measure one way Doppler shifts of astronomical objects, which may have radial speeds that are significant fractions of the speed of light.

2.5 Some relativistic Doppler calculations

This section may be skipped by those not interested in reading calculations. It contains nothing new, but working through multiple frames of reference gives some quite interesting insights.

For simplicity we shall use a transmitter moving at $\dot{\mathbf{x}}_{\mathsf{Tx}} = 0.6$ and a receiver moving at $\dot{\mathbf{x}}_{\mathsf{Rx}} = 0.8$, both relative to some arbitrary frame of reference (labeled with the subscript ref, as shown in figure 2.10). These values are 'friendly', since $\sqrt{1-0.6^2} = 0.8$ and $\sqrt{1-0.8^2} = 0.6$, nice round numbers!

Let the transmitted wavelength be $\lambda_{\mathsf{Tx}} = 0.1$ lightseconds (i.e., a frequency of 10 Hz, not quite an electromagnetic frequency, but then, a nice easy number to illustrate with). To set a baseline, we will first do the one way Doppler translation and do so through the reference frame.

The reference frame is effectively a 'receiver', which moves relative to the transmitter at -0.6.

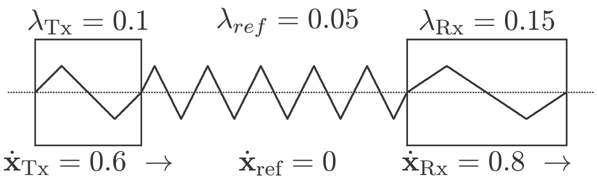

Figure 2.10: Relativistic Doppler shifts in wavelength, from the transmitter frame (left), through an arbitrary reference frame, to the receiver frame (right).

The transmitter wavelength translates to a reference frame wavelength of

$$\lambda_{ref} = 0.1\sqrt{\frac{1-0.6}{1+0.6}} = 0.05.$$

Then we translate from the reference frame to the actual receiver frame, which moves relative to the reference frame at speed 0.8c, giving

$$\lambda_{\mathsf{Rx}} = 0.05\sqrt{\frac{1+0.8}{1-0.8}} = 0.15.$$

This gives a relativistic Doppler shift from transmitter to receiver of

$$\frac{\Delta\lambda}{\lambda_{\mathsf{Tx}}} = \frac{0.15}{0.1} - 1 = 0.5,$$

a positive Doppler shift (i.e., a redshift) of half the transmitter wavelength.

Now, to compare, we will do a one-step transformation from transmitter to receiver, using the full relativistic equation. For this we need the relative speed of the receiver to the transmitter (relativistic subtraction of speed), which is

$$\dot{\mathbf{x}} = \frac{0.8-0.6}{1-0.8\times 0.6} \cong 0.3846,$$

giving a Doppler shift of

$$\frac{\Delta\lambda}{\lambda_{\mathsf{Tx}}} = \sqrt{\frac{1+0.3846}{1-0.3846}} - 1 = 0.5,$$

the same as for the two step transformation.

The two-step transformation may look like a way of determining the speed of the transmitter and the receiver relative to Newton's absolute frame of reference. Not so, because we chose an arbitrary reference frame, of which there are an infinite number.

There is also an infinite number of combinations of $\dot{\mathbf{x}}_{\mathsf{Tx}}$ and $\dot{\mathbf{x}}_{\mathsf{Rx}}$ that will give exactly the same Newtonian relative velocity and the same Newtonian Doppler shift. Special relativity dictates that a specific Doppler shift always implies one specific relative velocity.

Finally, let us look at the two way, radar type Doppler shift. We can do this in a one step calculation and find, quite simply

$$\frac{\Delta\lambda}{\lambda_{\mathsf{Tx}}} = \frac{1+0.3846}{1-0.3846} - 1 = 1.25,$$

meaning that the radar receiver will receive a wavelength of 1.25 times the transmitted frequency.

As a check on this value, we can work through the reference frame again, which gives a more intuitive 'feel' to the numbers. To this end, we have already done the first half of the calculations, i.e., up to the receiver (the $\lambda_{\mathsf{Rx}} = 0.15$ obtained above).

Now the 'target', reflecting the radar signal, is equivalent to a transmitter, moving at a speed 0.8 relative to the reference frame, which becomes the 'receiver', giving

$$\lambda_{ref_2} = 0.15\sqrt{\frac{1+0.8}{1-0.8}} = 0.45.$$

After this, we must go from the reference frame (which is equivalent to a transmitter) to the actual radar receiver. The reference frame is moving at speed -0.6 relative to the radar receiver, giving

$$\lambda_{\mathsf{Tx}} = 0.45\sqrt{\frac{1-0.6}{1+0.6}} = 0.225.$$

Since the original transmitted wavelength was 0.1, the total Doppler shift is

$$\frac{\Delta\lambda}{\lambda_{\mathsf{Tx}}} = \frac{0.225}{0.1} - 1 = 1.25,$$

the same as obtained through the one step calculation.

2.6 Summary

We have seen a fairly elaborate discussion of the synchronization of clocks, where we have seen how the absolute time frame was discarded through the relativity of simultaneity. This led to the Minkowski spacetime diagram where it was shown that the speed of light is indeed isotropic for all inertial observers in this representation.

The correlations between mass, energy and momentum from a special relativity viewpoint was discussed next. We saw that the rest mass of an object is essentially constant (unless it loses energy through radiation) and that one can view the moving mass to be more than the rest mass, because mass and energy is equivalent to each other.

The very important phenomenom Doppler shift was treated and the differences between Newtonian and relativistic Doppler shifts explained. We learned that one-way Newtonian Doppler shift depends on a static aether and that a specific Doppler shift can mean an infinite number of relative radial speeds.

Relativistic one-way Doppler shift is purely dependant upon the relative speed between the transmitter and receiver. We also learned that while one-way Doppler shifts differ markedly between the Newtonian and relativistic cases, the two-way Doppler shift shows less differences. The usual radar distance equation is simply an approximation for the low speed case.

We will now step off the purely inertial restriction and investigate how acceleration can be brought into special relativity. This is the topic of the next chapter.