Chapter 14

The Friedmann Equation

When the energy equation
seems
not to balance

14.1 A 'Newton cannon ball'

We will start this discussion by looking at an analogy with the familiar case of a cannon ball, shot up straight from the surface of the Earth.

If we ignore the drag from the atmosphere and influences from the Sun, the Moon and other planets, it is easy to write down the Newtonian equation of motion for the cannon ball. Radial velocity $\dot{r}$ changes against distance r as

$$\dot{r} = \pm\sqrt{\epsilon + 2GM/r}, \tag{14.1}$$

where ϵ is a constant proportional to the total (Newtonian) energy per unit mass of the cannon ball. Total energy is made up of positive kinetic energy and negative potential energy, so ϵ can be positive, zero or negative.

It is easy to see that $\epsilon = 0$ will give the standard escape speed $\dot{r}_e = \sqrt{2GM/r}$. A positive ϵ will produce a speed greater than the escape speed and visa versa for negative ϵ. So a cannon ball with $\epsilon \geq 0$ will escape from Earth and if $\epsilon < 0$, it will eventually fall back to Earth.

Now this looks suspiciously much like the case of the open universe, expanding forever, and the closed universe that will one day collapse back to it's origin. But does this simple Newtonian energy balance also hold for the expanding universe? The answer is: almost.

14.2 The Friedmann equation

During the 1920's, Friedmann showed that Einstein's field equations have a solution for an expanding universe that is extremely close to the 'Newton cannon ball' equation discussed above. In it's simplest form, the Friedmann equation is virtually indistinguishable from the Newton energy balance. The Friedmann equivalent* of eq 14.1 is

*This is not the 'real' Friedmann equation, which will be given shortly

$$\dot{R} = \pm\sqrt{-k + \frac{2GM}{R}}\ , \tag{14.2}$$

where M is the total mass of the universe and R the present radius of the entire universe (not the observable part). $\dot{R} = dR/dt$ is the 'velocity' of expansion and k is a 'curvature switch', selecting between open, flat and closed universes.

The $k = 0$ case is obviously the 'flat' universe. The $k = -1$ case is an open universe with negative curvature, but positive total energy. The $k = +1$ case is a closed universe with positive curvature, but negative total energy, meaning that the expansion will reverse sometime in the future.

Cosmologists do not work with the (unknown) mass of the universe in the Friedmann equation. They replace M with a function of the density (ρ) of the universe, i.e., $M = \frac{4}{3}\pi\rho R^3$, meaning multiplying the volume of the sphere ($\frac{4}{3}\pi R^3$) by it's mass density (ρ). Substituting M into eq. 14.2 gives the standard Friedmann equation

$$\dot{R} = \pm\sqrt{-k + \tfrac{8}{3}\pi G\rho R^2},$$

which can also be written as

$$\dot{a}/a = \pm\sqrt{-k/R^2 + \tfrac{8}{3}\pi G\rho}, \tag{14.3}$$

since $\dot{R}/R = \dot{a}/a$, where a is the expansion factor. Here $-k/R^2$ represents 'curvature density', with a sign depending on the value of k, and with a magnitude depending on the radius R.

14.3 The density parameter Ω

Cosmologists work with a dimensionless density parameter, defined as

$$\Omega = \frac{\rho}{\rho_c} = \frac{8\pi G}{3\hat{H}_0^2}\rho, \tag{14.4}$$

where ρ_c is a critical density that will produce a flat Einstein-de Sitter universe with $\Omega = 1$. For an open universe, $\Omega < 1$, which can be called an "under-dense" state and visa versa for a closed universe, which can be called "over-dense".

The Friedman equation can be worked into this most useful form [Peebles] at least for a 'matter only' universe:

$$\dot{a} = \bar{H}_0 \sqrt{1 - \Omega + \frac{\Omega}{a}}, \tag{14.5}$$

i.e., the expansion rate in terms of the Hubble constant, the density parameter and the expansion factor, all measurable quantities, at least in principle. It is mostly cast into the form

$$\frac{\dot{a}}{a} = \bar{H}_0 \sqrt{\frac{1 - \Omega}{a^2} + \frac{\Omega}{a^3}}, \tag{14.6}$$

because it is easier to understand—e.g., if the amount of matter remains constant, the matter density Ω is inversely proportional to the cube of the expansion factor, i.e., to the volume of the universe.

This is the expansion law for a 'curved' universe, where only matter density is accounted for. Note that when $\Omega = 1$, we have the expansion law of the Einstein-de Sitter model.

The quantity $1-\Omega$ is sometimes called the 'curvature parameter', i.e. $\Omega_R = 1 - \Omega$. (See e.g., [Peebles]). When $\Omega_R = 0$ there is no curvature and Ω_R goes positive and negative corresponding to positive or negative curvature.

Equation 14.5 was used to plot the curves in figure 14.1, not directly, but by numerically integrating dt with respect to a from $a = 0$ to $a = 1$. Since $\dot{a} = da/dt$, the integral is

$$t = \frac{1}{\bar{H}_0} \int_0^1 \frac{da}{\sqrt{1 - \Omega + \Omega/a}}. \tag{14.7}$$

A "middle of the range" Hubble constant of 64 km/s/Mpc were used.*

*It seems that at the time of writing, the 'best fit' value of H_0 is 72 km/s/Mpc.

The 'closed', 'flat' and 'open' curves tell us some important things. Firstly, the age of the universe as predicted by the standard cosmology can be read off the graphs for the three curves: about 8.6 Gy for the closed curve, 10 Gy for the flat curve and just over 12 Gy for the open curve, consistent with a Hubble constant of 64 km/s/Mps.

Secondly, all three curves cut the line $a(t) = 1$ with identical slopes. This must be so, because they were all calculated for the same H_0, which is the slope of the curves at $a(t) = 1$.

Thirdly, the rate of change of the slopes is the highest for the closed model, explaining why it is the one that will eventually return to $a(t) = 0$. The closed model starts expansion faster than the others, but being more dense, the expansion eventually stops and gravitational collapse follows.

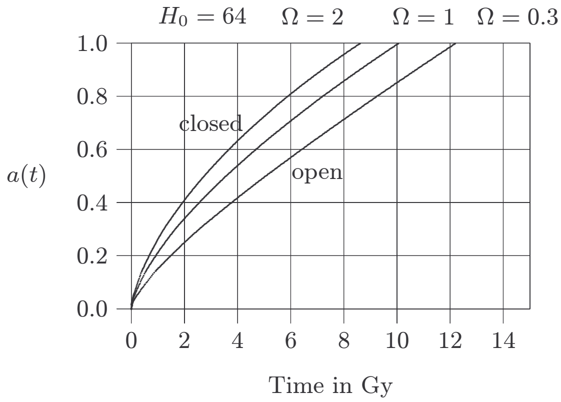

Figure 14.1: The expansion factor $a(t)$ against time for different values of Ω. The 'closed' curve is a section of an ellipse and the slope of the curve will eventually become negative and return to $a(t) = 0$, i.e., gravitational collapse. The 'flat' curve is a section of a parabola and the slope will tend towards zero as time tends to infinity. The 'open' curve is a section of a hyperbola and will have a positive slope forever.

14.4 Density accounting

At this point it is appropriate to introduce the fact that energy density need not necessarily be made op by mass alone. Since mass is the same thing as energy, all forms of energy must be accounted for. Cosmologists use three forms of energy in their "accounting": mass energy, radiation energy and vacuum energy. The respective contributions to Ω are denoted by Ω_m, Ω_r and Ω_v.

For the present value of Ω, the accountant's job is simple, because $\Omega = \Omega_m + \Omega_r + \Omega_v$. If however, we want to refer to the value of Ω at some other epoch, it is a bit more involved.

The density of matter decreases inversely proportional to the cube of the expansion factor (i.e., a^3). The radiation energy density decreases inversely proportional to a^4, being reduced by both the expanding volume and the redshift. Vacuum energy density remains constant because as more space is created by the expansion, so more vacuum energy can be created in a linear fashion.*

*That is if the vacuum energy is not exactly zero at all times.

This gives Ω as a function of the expansion factor

$$\Omega(a) = \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_v \; , \tag{14.8}$$

where Ω_m, Ω_r and Ω_v is the values of the present time. This illustrates the epoch dependency of Ω on the various factors making it up.

When the universe was very young and small ($a \ll 1$), the radiation component Ω_r/a^4 dominated the accounting. But since Ω_m is much larger than Ω_r, as soon as a grew larger, the mass component Ω_m/a^3 started to dominate. And if the vacuum energy density Ω_v has even a very small value, then as time goes on and $a \gg 1$, Ω_v will start to dominate the accounting.

We can now rewrite the Friedmann equation (eq. 14.6 above) in terms of the full Ω as

$$\dot{a}/a = \bar{H}_0 \sqrt{\frac{1-\Omega}{a^2} + \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_v} \; , \tag{14.9}$$

where $\Omega = \Omega_m + \Omega_r + \Omega_v$. Remember that all the Ω's have their present values, with the time dependencies sorted out by the a^n denominators.

Equation 14.9 is the general expansion law for most of the history of the universe.* Like before, the inverse of this equation can be numerically integrated to find the expansion curve for any makeup of Ω.

*It does not account for 'inflation', which is the topic of the next chapter.

All the above "accounting" for Ω seems to complicate matters considerably. And so it does, making it almost impossible to perceive how a "flat" $\Omega = 1$ can be maintained. Fortunately the present day contribution of radiation energy is extremely small. It was only in the first one ten-thousandths or so of the age of universe that radiation energy was important. After that, matter density dominated radiation density almost totally.

Vacuum energy is not as clear-cut as that, because as we have seen above, it does not get diluted by the expansion of the universe. The question is, does vacuum energy make any significant contribution to the total energy density, at least at the present time?

We will return to this question shortly, but first we will look at the question of the stability of the "flat" state, in other words, is $\Omega = 1$ a stable condition?

14.5 The stability of $\Omega = 1$

We have seen in the previous sections that for the universe to be flat, i.e. expanding just fast enough to prevent collapsing back into a singularity, the total of all contributions to the density parameter Ω must add up to unity. But to maintain $\Omega = 1$ needs a balancing act about as difficult as making a pencil stand on it's fine point on a flat table top. The smallest perturbation from unity will make Ω evolve away from that value as the universe expands.

Cosmologists are reasonably confident that the present value of Ω falls in the range 0.93 to 1.07. Then they calculate back to close to the big bang and find that Ω must have then been within a factor 1 in 10^{60} from unity then. If not, Ω would have evolved to lie outside the observational range today.

Because this is such an important aspect of cosmology, we will briefly examine why and how Ω evolves away from unity, once it deviates from that value.

A slightly imperfect analogy is the case of the cannon ball shot radially away from Earth where friction with the atmosphere is ignored. If the initial radial velocity is exactly equal to escape velocity $\sqrt{2GM/r}$, then the radial velocity will decrease, but it will always be equal to the escape velocity for the distance at which the cannon ball finds itself.

If the initial radial velocity was ever so slightly smaller than the escape velocity, the cannon ball will slow down too fast until it eventually starts to drop back to Earth again. In other words the velocity will eventually drastically diverge from the escape velocity.

For the universe at large, the same effect is at work. Slightly too small an expansion rate will cause the expansion rate to diverge away from the 'critical expansion rate' H_c, which can be expressed as follows:

$$H_c = \frac{\dot{a}_c}{a} = \sqrt{\frac{8\pi G}{3}\rho_c} \ . \tag{14.10}$$

An expansion rate slower than H_c will cause the expansion to stop and contraction to take over. An expansion rate higher than H_c will cause the density to drop too fast and the density parameter will diverge to zero.

Only if $H = H_c$ and $\Omega = 1$ will the critical balance be maintained indefinitely. It is however not possible to establish (by observation) whether $\Omega = 1$ precisely, due to inevitable uncertainties in the observations. Cosmologists come quite close, however.

14.6 The energy of the vacuum

The energy of the vacuum was first utilized by Einstein in his efforts to make his own equations of general relativity compatible with a static universe. Even standard Newton cosmology would not allow such a static universe, because the mutual attraction of all the matter in the universe would surely make it to contract. That is unless the universe as a whole is rotating relative to some or other absolute reference frame, which is apparently not the case.

The believe that the universe was static was however so strong in his time, that Einstein brought in a *cosmological constant* into his equations to act as a repulsive force, thus preventing his static universe from contracting.*

*The cosmological constant is proportional to the energy parameter of the vacuum: $\Lambda = 3\bar{H}_0^2\Omega_v$.

Later, when Hubble and others proved that the universe is actually expanding, Einstein repudiated the cosmological constant in his equations.

But Einstein's declaration that it was "the biggest blunder of his scientific career" did not make the cosmological constant go away completely. It was toyed with from time to time and today it is a hot topic in cosmology.

Why? Firstly, astronomers cannot find enough mass or other forms of energy to balance the expansion equations. Secondly, one of the best theories that explains how the universe got to be expanding in the first place, *inflation theory*, needs vacuum energy. Thirdly, it appears that the universal expansion is not slowing down as it "properly" should.

It may even be accelerating, if one takes recent observations at face value. We will return to inflation and possible acceleration in the expansion rate later, but first we need to get a feeling for what this *energy of the vacuum* is.

A "feeling" is about all that is within the scope of this book. Vacuum energy falls within the realms of quantum theory, which is a pretty complex subject.

The quantum vacuum is not an empty place. There are fluctuating quantum fields with all possible wavelengths that move in all possible directions. If averaged over time, these fields cancel out and we have a classical vacuum—resembling what we think of as empty space, with zero average energy.

If however the fields do not cancel out, we have, in the jargon of physics, a *false vacuum*. Over short periods of time, the quantum fields do not have to cancel out and if the resultant field is positive, then according to quantum theory, this positive field can act on matter as a repulsive cosmological force, something like "anti-gravity".

In an expanding universe, such a force extracts energy from the vacuum and converts it into additional kinetic energy of expansion. The vacuum must therefore end up with nett negative energy. This negative energy of the vacuum produces a contractive cosmological force, balancing the extra kinetic energy.

In this way, vacuum energy can work both ways (repulsive and contractive) at the same time. It is the negative component of the vacuum energy that is added into the 'accounting' equation for Ω (eq. 14.8 on page 185). In this way, the positive kinetic energy of expansion is precisely balanced by the negative contractive energy, maintaining a 'flat', $\Omega = 1$ universe.

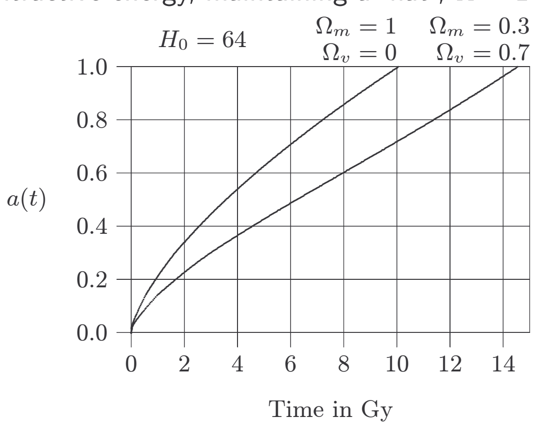

Figure 14.2: The expansion factor $a(t)$ against time for the $\Omega_m = 1$ and the $\Omega_m + \Omega_v = 1$ cases. The left curve has the same form as the 'flat' case in figure 14.1. Note how much older the universe with appreciable vacuum energy is—more than 14 Gy. ($H_0 = 64$ km/s/Mpc).

14.7 Summary of Friedmann model

In summary then, if the expansion energy accounting book does not balance, we have either an open or a closed universe. If the book balances, even if the rate of universal expansion is increasing due to vacuum energy, we have a 'flat', $\Omega = 1$ universe.

Recent observations seem to indicate that we live in such a flat universe, possibly dominated by vacuum energy already. Figure 14.2 shows a comparison between the expansion curves for a 'normal' flat universe and a flat universe with vacuum energy operating already.

It is reasonably assumed that $\Omega_r \approx 0$ today and that $\Omega = \Omega_m + \Omega_v = 1$. A Hubble constant of 64 km/s/Mpc was used, although the latest indications are that a value of 72 is the 'best fit'. This brings the age of the universe down to about 13.6 Gy.

In the next chapter we will see that vacuum energy could have been the mechanism that have started the whole process of universal expansion in the first place—the so called "inflationary big bang".